Dynamical Analogies

By

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PREFACE TO THE SECOND EDITION

The objective of *Dynamical Analogies* is to present the significant aspects of the subject within the framework of the existing state of this rapidly expanding science.

The first edition of this book, published in 1942, covered the subject matter of classical dynamical analogies among electrical, mechanical rectilineal, mechanical rotational and acoustical systems. The fundamental aspects of this area of engineering have not changed since that time. However, the subject matter of analogies has been extended into other related fields. In view of this expansion, a chapter on the magnetic analogy has been added, as well as two chapters on analogies in the fields of noise, distortion and feedback.

The mobility analogy has been found to be useful in engineering, particularly in the solution of mechanical rectilineal systems, because of the similarity between the diagrams in the mechanical and electrical systems. As a result, an enlarged chapter on the subject of the mobility analogy has been furnished to make the book complete.

Chapter I on "Introduction and Definitions" has been revised in the light of the new material contained in this edition. Chapter XI on "Applications" has been increased in scope.

The author wishes to express his appreciation to Miss Patricia Durnan for her work in typing the manuscript.

HARRY F. OLSON

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PREFACE TO THE FIRST EDITION

Analogies are useful for analysis in unexplored fields. By means of analogies an unfamiliar system may be compared with one that is better known. The relations and actions are more easily visualized, the mathematics more readily applied and the analytical solutions more readily obtained in the familiar system.

Although not generally so considered the electrical circuit is the most common and widely exploited vibrating system. By means of analogies the knowledge in electrical circuits may be applied to the solution of problems in mechanical and acoustical systems. In this procedure the mechanical or acoustical vibrating system is converted into the analogous electrical circuit. The problem is then reduced to the simple solution of an electrical circuit. This method has been used by acoustical engineers for the past twenty years in the development of all types of electro-acoustic transducers. Mechanical engineers have begun to use the same procedure for analyzing the action of mechanisms.

The importance and value of dynamical analogies to any one concerned with vibrating systems have led to a demand for expositions on this branch of dynamics. Accordingly this book has been written with the object of presenting the principles of dynamical analogies to the engineer.

This book deals with the analogies between electrical, mechanical rectilineal, mechanical rotational and acoustical systems. The subject matter is developed in stages from the simple element through to complex arrangements of multielement systems. As an aid in the establishment of these analogies a complete theme is depicted in each illustration.

The text assumes on the part of the reader a familiarity with the elements of alternating circuit theory and physics.

The author wishes to express his gratitude to his wife, Lorene E. Olson, for compilation and assistance in preparation and correction of the manuscript.

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HARRY F. OLSON

JANUARY, 1943

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CHAPTER I

INTRODUCTION AND DEFINITIONS

1.1. Introduction

A dynamical system is a system involving the motion of bodies and the action of forces in producing or changing the motion of the bodies. Dynamical systems may be classified as mechanical, electrical, acoustical, magnetic and electronic types that are found separately and in combinations. These various types of dynamical systems are incorporated in machines of all kinds for all manner of applications. For example, transportation, power, communication, education, agriculture, manufacturing, merchandising, business, clothing and housing all involve to a major extent the many and varied forms of dynamical systems. Thus it will be seen that dynamical systems embrace almost every facet of modern civilization.

The research, development, design, production and use of dynamical systems require an intimate knowledge of the theory, operation and application of machines employing dynamical systems. Analysis describing the performance of dynamical systems can be facilitated and implemented by employing the application of analogies. An analogy is a recognized relationship of consistent mutual similarity between the equations and structures appearing in two or more fields of knowledge, and an identification and association of the quantities and structural elements which play mutually similar roles in these equations or structures for the purpose of facilitating transfer of knowledge of mathematical or other procedures of analysis and behavior of the structure between these fields. Analogies are useful when it is desired to compare an unfamiliar system with one that is better known. The relations and actions are more easily visualized, the mathematics more readily applied and the analytical solutions more readily obtained in the familiar system. Analogies make it possible to extend the line of reasoning into unexplored fields.

The term *dynamical analogies* has been applied to the analysis of dynamical systems by the use of analogies. Specifically, dynamical analogies establish the analogies between electrical, mechanical, acoustical, magnetic and electronic systems.

A large part of engineering analysis is concerned with dynamical systems. Although not generally so considered, the electrical circuit is the most common example and the most widely exploited dynamical system. The equations of electrical circuit theory may be based on Maxwell's dynamical theory, in which the currents play the role of velocities. Expressions for kinetic energy, potential energy and dissipation show that network equations are deducible from general dynamic equations. In other words, an electrical circuit may be considered to be a dynamical system. This immediately suggests analogies between electrical circuits and other dynamical systems, as for example, mechanical and acoustical dynamical systems.

The equations of motion of mechanical systems were developed a long time before any attention was given to equations for electrical circuits. For this reason, in the early days of electrical circuit theory, it was natural to explain the action in terms of mechanical phenomena. However, at the present time electrical circuit theory has been developed to a much higher state than the corresponding theory of mechanical systems. The number of engineers and scientists versed in electrical circuit theory is many times the number equally familiar with mechanical systems.

Almost any work involving mechanical or acoustical systems also includes electrical systems and electrical circuit theory. The acoustical engineer is interested in sound reproduction or the conversion of electrical or mechanical energy into acoustical energy, the development of vibrating systems and the control of sound vibrations. This involves acoustical, electroacoustical, mechanoacoustical or electromechanoacoustical systems. The mechanical engineer is interested in the development of various mechanisms or vibrating systems involving masses, springs and friction.

Electrical circuit theory is the branch of electromagnetic theory which deals with electrical oscillations in linear electrical networks.[1] An elec-

[1] The use of the terms "circuit" and "network" in the literature is not established. The term "circuit" is often used to designate a network with several branches.

trical network is a connected set of separate circuits termed branches or meshes. A circuit may be defined as a physical entity in which varying magnitudes may be specified in terms of time and a single dimension.[2] The branches or meshes are composed of elements, the constituent parts of a circuit. Electrical elements are resistance, inductance and capacitance. Vibrations in one dimension occur in mechanical systems made up of mechanical elements, as for example, various assemblies of masses, springs and brakes. Acoustical systems in which the dimensions are small compared to the wavelength are vibrations in a single dimension.

The number of independent variables required to completely specify the motion of every part of a vibrating system is a measure of the number of degrees of freedom of the system. If only a single variable is needed the system is said to have a single degree of freedom. In an electrical circuit the number of degrees of freedom is equal to the number of independent closed meshes or circuits.

The use of complex notation has been applied extensively to electrical circuits. Of course, this operational method can be applied to any analytically similar system.

Mathematically the elements in an electrical network are the coefficients in the differential equations describing the network. When the electric circuit theory is based upon Maxwell's dynamics, the network forms a dynamical system in which the currents play the role of velocities. In the same way the coefficients in the differential equations of a mechanical or acoustical system may be looked upon as mechanical or acoustical elements. Kirchhoff's electromotive force law plays the same role in setting up the electrical equations as D'Alembert's principle does in setting up the mechanical and acoustical equations. That is to say, every electrical, mechanical or acoustical system may be considered as a combination of electrical, mechanical or acoustical elements. Therefore, any mechanical or acoustical system may be reduced to an electrical network and the problem may be solved by electrical circuit theory.

In view of the tremendous amount of study which has been directed towards the solution of circuits, particularly electrical circuits, and the engineer's familiarity with electrical circuits, it is logical to apply this

[2] The term "single dimension" implies that the movement or variation occurs along a path. In a field problem there is variation in two or three dimensions.

knowledge to the solution of vibration problems in other fields by the same theory as that used in the solution of electrical circuits.

In this book, the author has attempted to outline the essentials of dynamical analogies from the standpoint of the engineer or applied scientist. Differential equations are used to show the basis for the analogies between electrical, mechanical and acoustical systems. However, the text has been written and illustrated so that the derivations may be taken for granted. The principal objective in this book is the establishment of analogies between electrical, mechanical and acoustical systems so that any one familiar with electrical circuits will be able to analyze the action of vibrating systems.

The classical impedance analogies which will be presented in this book are formal ones due to the similarity of the differential equations of electrical, mechanical and acoustical systems. This does not imply that there is any physical similarity between quantities occupying the same position in their respective equations. Therefore, the impedance analogies are not the only ones possible of development for useful applications of dynamical analogies. For example, in the past, mechanical impedance has been defined by some authors—in addition to that as the ratio of force to velocity as will be developed in this book—as the ratio of pressure to velocity, the ratio of force to displacement and the ratio of pressure to displacement. In this connection a useful analogy, termed the *mobility analogy*, has been employed on a wide scale to solve problems in mechanical vibrating systems. In the mobility analogy, mechanical mobility is defined as the complex ratio of velocity to force. The mobility analogy will be considered in Chapter 14.

1.2. Definitions

A few of the terms [2] used in dynamical analogies will be defined in this section. Terms not listed below will be defined in subsequent sections.

Periodic Quantity. An oscillating quantity, the values of which recur for equal increments of the independent variable. If a periodic quantity y is a function of x, then y has the property that $y = f(x) = f(x + T)$, where T, a constant, is a period of y. The smallest positive value of T

[2] Approximately one-half of the definitions in this chapter are taken from the American Standards Association standards. The remainder, which have not been defined at this time by any standards group, are written to conform with the analogous existing standards.

is the primitive period of y, generally called simply the period of y. In general a periodic function can be expanded into a series of the form.

$$y = f(x) = A_0 + A_1 \sin(\omega x + \alpha_1) + A_2 \sin(2\omega x + \alpha_2) + \ldots,$$

where ω, a positive constant, equals 2π divided by the period T, and the A's and α's are constants which may be positive, negative, or zero. This is called a Fourier series.

Cycle. One complete set of the recurrent values of a periodic quantity.

Period. The time required for one cycle of a periodic quantity. The unit is the second.

Frequency. The number of cycles occurring per unit of time, or which would occur per unit of time if all subsequent cycles were identical with the cycle under consideration. The frequency is the reciprocal of the period. The unit is the cycle per second.

Octave. The interval between two frequencies having a ratio of two to one.

Fundamental Frequency. The lowest component frequency of a periodic quantity.

Harmonic. A component of a periodic quantity which is an integral multiple of the fundamental frequency. For example, a component the frequency of which is twice the fundamental frequency is called the second harmonic.

Basic Frequency. The frequency of a periodic quantity which is considered to be the most important. In a driven system it would in general be the driving frequency while in most periodic waves it would correspond to the fundamental frequency.

Subharmonic. A component of a periodic quantity having a frequency which is an integral submultiple of the basic frequency. NOTE: The term "subharmonic" is generally applied in the case of a driven system whose vibration has frequency components of lower frequency than the driving frequency.

Wave. A propagated disturbance, usually a periodic quantity in an electrical, mechanical or acoustical system.

Wavelength of a periodic wave in an isotropic medium is the perpendicular distance between two wave fronts in which the displacements have a phase difference of one complete cycle.

Response of a system, machine or transducer is the motion or other output resulting from the excitation or stimulus.

Excitation or *Stimulus.* An external force or other input applied to a system, machine or transducer which causes the system, machine or transducer to respond in some way.

Displacement. A vector quantity which specifies the change of position of a body or particle, and is usually measured from the mean position or position of rest. The unit is the centimeter.

Velocity. A vector quantity which specifies the time rate of change of displacement with respect to a reference frame. The unit is the centimeter per second.

Acceleration. A vector quantity which specifies the time rate of change of velocity. The unit is the centimeter per second per second.

Vibration. The variation with time of the magnitude of a quantity, generally a parameter which defines the motion of a dynamical system, with respect to some reference when the magnitude is alternately greater and smaller than the reference.

Steady State Vibration exists in a system if the velocity of each particle is a periodic quantity.

Angular Frequency of a periodic quantity, in radians per unit of time, is the frequency multiplied by 2π.

Damping. The dissipation of energy in an oscillating system.

Electromotive Force around a closed path is the work required to carry a small positive charge around that path divided by the charge. The electromotive force is in abvolts, the work in ergs, and the charge in abcoulombs.

Abvolt. The unit of electromotive force.

Instantaneous Electromotive Force between two points is the total instantaneous electromotive force. The unit is the abvolt.

Effective Electromotive Force. The root mean square of the instantaneous electromotive force over a complete cycle between two points. The unit is the abvolt.

Maximum Electromotive Force. The maximum absolute value of the instantaneous electromotive force during that cycle. The unit is the abvolt.

Peak Electromotive Force for any specified time interval is the maximum absolute value of the instantaneous electromotive force during that interval. The unit is the abvolt.

Momentum of a particle or body is the product of its mass and velocity. The unit is the gram centimeter per second.

Force. The interaction between systems which has vector characteristics and a magnitude which can be measured by the acceleration which the force gives to a defined mass. The force acting upon a particle or a body is assumed as the cause of the acceleration of the particle or body. Force is the vector function, which, in magnitude and direction, equals the time rate of change of momentum of the particle or body. The unit is the dyne.

Dyne. The unit of force or mechanomotive force.

Instantaneous Force (*Instantaneous Mechanomotive Force*) at a point is the total instantaneous force. The unit is the dyne.

Effective Force (*Effective Mechanomotive Force*). The root mean square of the instantaneous force over a complete cycle. The unit is the dyne.

Maximum Force (*Maximum Mechanomotive Force*). The maximum absolute value of the instantaneous force during that cycle. The unit is the dyne.

Peak Force (*Peak Mechanomotive Force*) for any specified interval is the maximum absolute value of the instantaneous force during that interval. The unit is the dyne.

Angular Momentum of a particle or body is the total moment of momentum of a particle or body with respect to some origin of coordinates. The unit is in dyne (centimeter)^2 per second.

Torque (*Moment of Force*) acting upon a particle or body is assumed as the cause of the angular acceleration of the particle or body. Torque equals the time rate of change of angular momentum of a particle or body. The unit is the dyne centimeter.

Dyne Centimeter. The unit of torque or rotatomotive force.

Instantaneous Torque (*Instantaneous Rotatomotive Force*) at a point is the total instantaneous torque. The unit is the dyne centimeter.

Effective Torque (*Effective Rotatomotive Force*). The root mean square of the instantaneous torque over a complete cycle. The unit is the dyne centimeter.

Maximum Torque (*Maximum Rotatomotive Force*). The maximum absolute value of the instantaneous torque during that cycle. The unit is the dyne centimeter.

Peak Torque (*Peak Rotatomotive Force*) for a specified interval is the maximum absolute value of the instantaneous torque during that interval. The unit is the dyne centimeter.

Pressure at a point in the medium is the force exerted over a surface

divided by the area of the surface in the specified region. The unit is the dyne per square centimeter.

Sound Pressure. An alteration in the pressure in an elastic medium. The unit is the dyne per square centimeter.

Dyne per Square Centimeter. The unit of sound pressure.

Static Pressure. The pressure that would exist in a medium with no sound waves present. The unit is the dyne per square centimeter.

Instantaneous Sound Pressure (*Instantaneous Acoustomotive Force*) at a point is the total instantaneous pressure at the point minus the static pressure. The unit is the dyne per square centimeter.

Effective Sound Pressure (*Effective Acoustomotive Force*) at a point is the root mean square value of the instantaneous sound pressure over a complete cycle at the point. The unit is the dyne per square centimeter.

Maximum Sound Pressure (*Maximum Acoustomotive Force*) for any given cycle is the maximum absolute value of the instantaneous sound pressure during that cycle. The unit is the dyne per square centimeter.

Peak Sound Pressure (*Peak Acoustomotive Force*) for any specified time interval is the maximum absolute value of the instantaneous sound pressure in that interval. The unit is the dyne per square centimeter.

Current. The rate of flow of positive electrical charges. One abampere transports one abcoulomb of charge per second. The unit is the abampere.

Abampere. The unit of current.

Instantaneous Current at a point is the total instantaneous current at that point. The unit is the abampere.

Effective Current at a point is the root mean square value of the instantaneous current over a complete cycle at that point. The unit is the abampere.

Maximum Current for any given cycle is the maximum absolute value of the instantaneous current during that cycle. The unit is the abampere.

Peak Current for any specified time interval is the maximum absolute value of the instantaneous current in that interval. The unit is the abampere.

Velocity of a point is the time rate of change of a position vector of that point with respect to the inertial frame. The unit is the centimeter per second.

Centimeter per Second. The unit of velocity.

Instantaneous Velocity at a point is the total instantaneous velocity at that point. The unit is the centimeter per second.

Effective Velocity at a point is the root mean square value of the instantaneous velocity over a complete cycle at that point. The unit is the centimeter per second.

Maximum Velocity for any given cycle is the maximum absolute value of the instantaneous velocity during that cycle. The unit is the centimeter per second.

Peak Velocity for any specified time interval is the maximum absolute value of the instantaneous velocity in that interval. The unit is the centimeter per second.

Angular Velocity of a plane is the time rate of change of the angular position vector of that plane with respect to the inertial frame. The unit is the radian per second.

Radian per Second. The unit of angular velocity.

Instantaneous Angular Velocity at a point is the total instantaneous angular velocity at that point. The unit is the radian per second.

Effective Angular Velocity at a point is the root mean square value of the instantaneous angular velocity over a complete cycle at the point. The unit is the radian per second.

Maximum Angular Velocity for any given cycle is the maximum absolute value of the instantaneous angular velocity during that cycle. The unit is the radian per second.

Peak Angular Velocity for any specified time interval is the maximum absolute value of the instantaneous angular velocity in that interval. The unit is the radian per second.

Volume Current (*Volume Velocity*). The rate of flow of the medium through a specific area. The unit is the cubic centimeter per second.

Cubic Centimeter per Second. The unit of volume current.

Instantaneous Volume Current at a point is the total instantaneous volume current at that point. The unit is the cubic centimeter per second.

Effective Volume Current at a point is the root mean square value of the instantaneous volume current over a complete cycle at that point. The unit is the cubic centimeter per second.

Maximum Volume Current for any given cycle is the maximum absolute value of the instantaneous volume current during that cycle. The unit is the cubic centimeter per second.

Peak Volume Current for any specified time interval is the maximum absolute value of the instantaneous volume current in that interval. The unit is the cubic centimeter per second.

Electrical Impedance. The complex quotient of the alternating electromotive force applied to the system by the resulting current. The unit is the abohm.

Electrical Abohm. An electrical resistance, reactance or impedance is said to have a magnitude of one abohm when an electromotive force of one abvolt produces a current of one abampere.

Electrical Resistance. The real part of the electrical impedance. This is the part responsible for the dissipation of energy. The unit is the abohm.

Electrical Reactance. The imaginary part of the electrical impedance. The unit is the abohm.

Inductance in an electrical system is that coefficient which, when multiplied by 2π times the frequency, gives the positive imaginary part of the electrical impedance. The unit is the abhenry.

Electrical Capacitance in an electrical system is that coefficient which, when multiplied by 2π times the frequency, is the reciprocal of the negative imaginary part of the electrical impedance. The unit is the abfarad.

Mechanical Rectilineal Impedance[4] (*Mechanical Impedance*). The complex quotient of the alternating force applied to the system by the resulting linear velocity in the direction of the force at its point of application. The unit is the mechanical ohm.

Mechanical Ohm. A mechanical rectilineal resistance, reactance or impedance is said to have a magnitude of one mechanical ohm when a force of one dyne produces a velocity of one centimeter per second.

Mechanical Rectilineal Resistance (*Mechanical Resistance*). The real part of the mechanical rectilineal impedance. This is the part responsible for the dissipation of energy. The unit is the mechanical ohm.

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[4] The word "mechanical" is ordinarily used as a modifier to designate a mechanical system with rectilineal displacements and the word "rotational" is ordinarily used as a modifier to designate a mechanical system with rotational displacements. To avoid ambiguity in this book, where both systems are considered concurrently, the words "mechanical rectilineal" are used as modifiers to designate a mechanical system with rectilineal displacements and the words "mechanical rotational" are used as modifiers to designate a mechanical system with rotational displacements.

Mechanical Rectilineal Reactance (*Mechanical Reactance*). The imaginary part of the mechanical rectilineal impedance. The unit is the mechanical ohm.

Mass in a mechanical system is that coefficient which, when multiplied by 2π times the frequency, gives the positive imaginary part of the mechanical rectilineal impedance. The unit is the gram.

Compliance in a mechanical system is that coefficient which, when multiplied by 2π times the frequency, is the reciprocal of the negative imaginary part of the mechanical rectilineal impedance. The unit is the centimeter per dyne.

Mechanical Rotational Impedance[6] (*Rotational Impedance*). The complex quotient of the alternating torque applied to the system by the resulting angular velocity in the direction of the torque at its point of application. The unit is the rotational ohm.

Rotational Ohm. A mechanical rotational resistance, reactance or impedance is said to have a magnitude of one rotational ohm when a torque of one dyne centimeter produces an angular velocity of one radian per second.

Mechanical Rotational Resistance (*Rotational Resistance*). The real part of the mechanical rotational impedance. This is the part responsible for the dissipation of energy. The unit is the rotational ohm.

Mechanical Rotational Reactance (*Rotational Reactance*). The imaginary part of the mechanical rotational impedance. The unit is the rotational ohm.

Moment of Inertia in a mechanical rotational system is that coefficient which, when multiplied by 2π times the frequency, gives the positive imaginary part of the mechanical rotational impedance. The unit is the gram centimeter to the second power.

Rotational Compliance in a mechanical rotational system is that coefficient which, when multiplied by 2π times the frequency, is the reciprocal of the negative imaginary part of the mechanical rotational impedance. The unit is the radian per centimeter per dyne.

Acoustical Impedance. The complex quotient of the pressure applied to the system by the resulting volume current. The unit is the acoustical ohm.

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[6] See footnote 4, page 10.

Acoustical Ohm. An acoustical resistance, reactance or impedance is said to have a magnitude of one acoustical ohm when a pressure of one dyne per square centimeter produces a volume current of one cubic centimeter per second.

Acoustical Resistance. The real part of the acoustical impedance. This is the part responsible for the dissipation of energy. The unit is the acoustical ohm.

Acoustical Reactance. The imaginary part of the acoustical impedance. The unit is the acoustical ohm.

Inertance in an acoustical system is that coefficient which, when multiplied by 2π times the frequency, gives the positive imaginary part of the acoustical impedance. The unit is the gram per centimeter to the fourth power.

Acoustical Capacitance (Acoustical Compliance) in an acoustical system is that coefficient which, when multiplied by 2π times the frequency, is the reciprocal negative imaginary part of the acoustical impedance. The unit is the centimeter to the fifth power per dyne.

Element[6] or *Circuit Parameter* in an electrical system defines a distinct activity in its part of the circuit. In the same way, an element in a mechanical rectilineal, mechanical rotational or acoustical system defines a distinct activity in its part of the system. The elements in an electrical circuit are electrical resistance, inductance and electrical capacitance. The elements in a mechanical rectilineal system are mechanical rectilineal resistance, mass and compliance. The elements in a mechanical rotational system are mechanical rotational resistance, moment of inertia, and rotational compliance. The elements in an acoustical system are acoustical resistance, inertance and acoustical capacitance.

System. An assemblage of elements united by some form of regular interaction or interdependence.

Electrical System. A system adapted for the transmission of electrical currents consisting of one or all of the electrical elements: electrical resistance, inductance and electrical capacitance.

Mechanical Rectilineal System. A system adapted for the transmission of vibrations consisting of one or all of the following mechanical rectilineal elements: mechanical rectilineal resistance, mass and compliance.

[6] Elements are defined and described in Chapter II.

Mechanical Rotational System. A system adapted for the transmission of rotational vibrations consisting of one or all of the following mechanical rotational elements: mechanical rotational resistance, moment of inertia and rotational compliance.

Acoustical System. A system adapted for the transmission of sound consisting of one or all of the following acoustical elements: acoustical resistance, inertance and acoustical capacitance.

Work. The accomplishment of a change in the complexion of a system against an opposing force. Work is the transference of energy by some process. The unit is the erg.

Power. The rate of transfer of energy. Power is the rate of doing work. The unit is the dyne centimeter per second.

Energy. The capacity for performing work. Energy is stored work. The unit is the dyne centimeter.

Kinetic Energy of a system is due to motion of a part or parts of the system with reference to other parts. The unit is the dyne centimeter. See Sect. 3.3.

Potential Energy of a system is due to stresses between different parts of the system. The unit is the dyne centimeter. See Sect. 3.3.

Transducer. A device actuated by power from one system and supplying power in the same or any other form to a second system. Either of these systems may be electrical, mechanical or acoustical.

Machine. A system consisting of two or more resistant, relatively-constrained parts, which may serve to transmit and modify force and motion so as to do some desired kind of work, and a complex combination of such parts.

Transmission System. A system which transmits power, voltage, current, force, velocity, torque, angular velocity, pressure or volume current.

Transmission in a system refers to the transmission of power, voltage, current, force, velocity, torque, angular velocity, pressure or volume current.

Signal. A piece of intelligence, message, wave or effect conveyed in a transmission system.

Passive System, Machine or *Transducer.* A system, machine or transducer in which all the power delivered to the load is obtained from the power accepted by the system, machine or transducer from the source.

Noise Level. The level of noise. The type of noise must be indicated by a further modifier.

Overload Level of a machine, transducer, component, element or system is that level at which operation ceases to be satisfactory as a result of distortion in the output, overheating and so forth.

Transmission Loss (*or Gain*). The transmission loss due to a system joining a load having a given electrical, mechanical rectilineal, mechanical rotational or acoustical impedance and a source having a given electrical, mechanical rectilineal, mechanical rotational or acoustical impedance and a given electromotive force, force, torque or pressure is expressed by the logarithm of the ratio of the power delivered to the load to the power delivered to the load under some reference condition. For a loss the reference power is greater. For a gain the reference power is smaller.

Decibel. The abbreviation db is used for the decibel. The bel is the fundamental division of a logarithmic scale expressing the ratio of two *amounts of power, the number of bels denoting such a ratio being the* logarithm to the base ten of this ratio. The decibel is one-tenth of a bel. For example, with P_1 and P_2 designating two amounts of power and n the number of decibels denoting their ratio

$$n = 10 \log_{10} \frac{P_1}{P_2}, \text{ decibels}$$

When the conditions are such that ratios of voltages or ratios of currents (or analogous quantities such as forces or velocities, torques or angular velocities, pressures or volume currents) are the square roots of the corresponding power ratios, the number of decibels by which the corresponding powers differ is expressed by the following formulas:

$$n = 20 \log_{10} \frac{i_1}{i_2}, \text{ decibels}$$

$$n = 20 \log_{10} \frac{e_1}{e_2}, \text{ decibels}$$

where i_1/i_2 and e_1/e_2 are the given current and voltage ratios respectively.

Power Level Gain. The amount by which the output power level in decibels exceeds the input power.

Rate of Decay. The time rate at which the level (power, voltage, pressure, force and so forth) is decreasing at a given point and at a given time. The rate is the decibel per second.

Relaxation Time. The time taken by an exponentially decaying oscillation to decrease in amplitude by a factor of $1/\epsilon = 0.3679$.

Lagrange's Equations. A set of equations of motion for a dynamical system. For a conservative system the equation of motion for the nth particle is

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_n} - \frac{\partial T}{\partial \dot{q}_n} = 0$$

where T is the Lagrangian function or kinetic potential, q_n is the generalized coordinate of the nth particle ($n = 1, 2, 3 \ldots n$, where n is the number of degrees of freedom of the system), $\dot{q}_n$ is the generalized velocity of the nth particle. See Sect. 4.6.

Newton's Laws. The dynamics of particles located in an inertial frame of reference is governed by Newton's three laws of motion as follows:

1. A particle, not under the action of a force, will maintain its velocity unchanged in magnitude and direction.
2. A force acting on a particle causes a change of momentum of the particle, the rate of change of momentum being vectorially equal to the force.
3. If one particle exerts a force on a second, then the second exerts a force, equal in magnitude but opposite in direction, on the first.

D'Alembert's Principle. The algebraic sum of the forces applied to a particle are zero. See Sect. 3.8.

Conservation of Momentum. If the total force acting upon a particle is zero, the momentum of the particle is a constant.

Conservation of Energy. If the total force acting upon a particle is conservative, the sum of the kinetic and potential energies is a constant.

Faraday's Law of Induction. The electromotive force induced in an electrical circuit is proportional to the rate of change of the magnetic flux linking the circuit.

Joule's Law. The rate of change of the production of heat in a constant-resistance electrical circuit is proportional to the square of the current.

Lenz's Law. The current induced in a circuit due to a change in the magnetic flux through it or to its motion in a magnetic field is so directed

as to oppose the change in flux or to exert a mechanical force opposing the motion.

Ohm's Law. The current in an electric circuit is directly proportional to the electromotive force in the circuit.

Kirchhoff's Laws. The algebraic sum of the currents flowing into any point in a network is zero. The algebraic sum of the products of current by resistance around any closed path in a network equals the algebraic sum of the electromotive forces in that path. See Sect. 3.8.

1.3. Publications

Dynamical analogies have been developed and used for a period of more than three decades. During this time there have been many publications reporting the developments in this field. A list of these publications, in their reverse chronological order, follows:

Firestone, *American Institute of Physics Handbook*, "The Mobility and Classical Impedance Analogies," McGraw-Hill Book Co., New York, N. Y., 1957.

Olson, *American Institute of Physics Handbook*, "Classical Electro-Dynamical Analogies," McGraw-Hill Book Co., New York, N. Y., 1957.

Olson, "Acoustical Engineering," D. Van Nostrand Co., Princeton, N. J., 1957.

Firestone, Floyd A., *Jour. Acous. Soc. Amer.*, Vol. 28, No. 6, p. 1117, 1956.

Trent, H. M., *Jour. Acous. Soc. Amer.*, Vol. 27, No. 3, p. 500, 1955.

Bauer, B. B., *Jour. Acous. Soc. Amer.*, Vol. 25, No. 5, p. 837, 1953.

Raymond, F., *Rev. gen. elec.*, Vol. 61, No. 10, p. 465, 1952.

LeCorbelier and Yueng, *Jour. Acous. Soc. Amer.*, Vol. 24, No. 6, p. 643, 1952.

Mason, "Electromechanical Transducers and Wave Filters," D. Van Nostrand Co., Princeton, N. J., 1948.

Bloch, A., *J. Inst. Elec. Eng.* (London) Vol. 92, Pt. 1, p. 157, 1945.

Olson, "Dynamical Analogies," D. Van Nostrand Co., Princeton, N. J., 1943.

Firestone, Floyd A., *Jour. Appli. Phys.*, Vol. 9, No. 5, p. 373, 1938.

Firestone, Floyd A., *Jour. Acous. Soc. Amer.*, Vol. 4, No. 3, p. 249, 1933.

Hahnle, Walter, *Wissen. Veroff Siemens-Konzern*, Vol. 11, p. 1, 1932.

Darrieus, M., *Bull. Soc. Franc. Elec.*, Vol. 96, p. 794, 1929.

CHAPTER II

ELEMENTS

2.1. Introduction

An element or circuit parameter in an electrical system defines a distinct activity in its part of the circuit. In an electrical system these elements are resistance, inductance and capacitance. They are distinguished from the devices; resistor, inductor and capacitor. A resistor, inductor and capacitor idealized to have only resistance, inductance and capacitance is a circuit element. As indicated in the preceding chapter, the study of mechanical and acoustical systems is facilitated by the introduction of elements analogous to the elements of an electric circuit. In this procedure, the first step is to develop the elements in these vibrating systems. It is the purpose of this chapter to define and describe electrical, mechanical rectilineal, mechanical rotational and acoustical elements.[1]

2.2. Resistance

A. *Electrical Resistance.*—Electrical energy is changed into heat by the passage of an electrical current through a resistance. Energy is lost by the system when a charge q is driven through a resistance by a voltage e. Resistance is the circuit element which causes dissipation.

Electrical resistance r_E, in abohms, is defined as

$$r_E = \frac{e}{i} \qquad 2.1$$

where e = voltage across the resistance, in abvolts, and
i = current through the resistance, in abamperes.

Equation 2.1 states that the electromotive force across an electrical resistance is proportional to the electrical resistance and the current.

[1] See footnote 4, page 10.

B. *Mechanical Rectilineal Resistance.*—Mechanical rectilineal energy is changed into heat by a rectilinear motion which is opposed by linear resistance (friction). In a mechanical system dissipation is due to friction. Energy is lost by the system when a mechanical rectilineal resistance is displaced a distance x by a force f_M.

Mechanical rectilineal resistance (termed mechanical resistance) r_M, in mechanical ohms, is defined as

$$r_M = \frac{f_M}{u} \qquad 2.2$$

where f_M = applied mechanical force, in dynes, and
u = velocity at the point of application of the force, in centimeters per second.

Equation 2.2 states that the driving force applied to a mechanical rectilineal resistance is proportional to the mechanical rectilineal resistance and the linear velocity.

C. *Mechanical Rotational Resistance.*—Mechanical rotational energy is changed into heat by a rotational motion which is opposed by a rotational resistance (rotational friction). Energy is lost by the system when a mechanical rotational resistance is displaced by an angle ϕ by a torque f_R.

Mechanical rotational resistance (termed rotational resistance) r_R, in rotational ohms, is defined as

$$r_R = \frac{f_R}{\theta} \qquad 2.3$$

where f_R = applied torque, in dyne centimeters, and
θ = angular velocity at the point of application about the axis, in radians per second.

Equation 2.3 states that the driving torque applied to a mechanical rotational resistance is proportional to the mechanical rotational resistance and the angular velocity.

D. *Acoustical Resistance.*—In an acoustical system dissipation may be due to the fluid resistance or radiation resistance. At this point the former type of acoustical resistance will be considered. Acoustical energy is changed into heat by the passage of a fluid through an acoustical resistance. The resistance is due to viscosity. Energy is lost by the system when a volume X is driven through an acoustical resistance by a pressure p.

Acoustical resistance r_A, in acoustical ohms, is defined as

$$r_A = \frac{p}{U} \qquad 2.4$$

where p = pressure, in dynes per square centimeter, and
U = volume current, in cubic centimeters per second.

Equation 2.4 states that the driving pressure applied to an acoustical resistance is proportional to the acoustical resistance and the volume current.

The transmission of sound waves or direct currents of air through small constrictions is primarily governed by acoustical resistance due to viscosity. A tube of small diameter, a narrow slit, and metal, or cotton or silk cloth are a few examples of systems which exhibit acoustical resistance. There is also, in addition to the resistive component, a reactive component. However, the ratio of the two components is a function of the dimensions. This is illustrated by the following equation for the acoustic impedance [2] of a narrow slit.

$$z_A = \frac{12\mu w}{d^3 l} + j\frac{6\rho w\omega}{5ld} \qquad 2.5$$

where μ = viscosity coefficient, 1.86×10^{-4} for air, density, in grams per cubic centimeter,
d = thickness of the slit normal to the direction of flow, in centimeters,
l = width of the slit normal to the direction of flow, in centimeters,
w = length of the slit in the direction of flow, in centimeters,
$\omega = 2\pi f$, and
f = frequency in cycles per second.

Any ratio of acoustical resistance to acoustical reactance can be obtained by a suitable value of d. Then the value of acoustical resistance can be obtained by an appropriate value of w and l. The same expedient may be employed in the case of any acoustical resistance in which the resistance is due to viscosity.

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[2] Olson, "Acoustical Engineering," D. Van Nostrand Co., Princeton, N. J., 1957.

2.3. Inductance, Mass, Moment of Inertia, Inertance

A. *Inductance.*—Electromagnetic energy is associated with inductance. Electromagnetic energy increases as the current in the inductance increases. It decreases when the current decreases. It remains constant when the current in the inductance is a constant. Inductance is the electrical circuit element which opposes a change in current. Inductance L, in abhenries, is defined as

$$e = L\frac{di}{dt} \tag{2.6}$$

where e = electromotive or driving force, in abvolts, and
di/dt = rate of change of current, in abamperes per second.

Equation 2.6 states that the electromotive force across an inductance is proportional to the inductance and the rate of change of current.

B. *Mass.*—Mechanical rectilineal inertial energy is associated with mass in the mechanical rectilineal system. Mechanical rectilineal energy increases as the linear velocity of a mass increases, that is, during linear acceleration. It decreases when the velocity decreases. It remains constant when the velocity is a constant. Mass is the mechanical element which opposes a change of velocity. Mass m, in grams, is defined as

$$f_M = m\frac{du}{dt} \tag{2.7}$$

where du/dt = acceleration, in centimeters per second per second, and
f_M = driving force, in dynes.

Equation 2.7 states that the driving force applied to the mass is proportional to the mass and the rate of change of linear velocity.

C. *Moment of Inertia.*—Mechanical rotational inertial energy is associated with moment of inertia in the mechanical rotational system. Mechanical rotational energy increases as the angular velocity of a moment of inertia increases, that is, during angular acceleration. It decreases when the angular velocity decreases. It remains a constant when the angular velocity is a constant. Moment of inertia I, in gram (centimeter)^2, is given by

$$f_R = I\frac{d\theta}{dt} \tag{2.8}$$

where $d\theta/dt$ = angular acceleration, in radians per second per second, and

f_R = torque, in dyne centimeters.

Equation 2.8 states that the driving torque applied to the moment of inertia is proportional to the moment of inertia and the rate of change of angular velocity.

D. *Inertance.*—Acoustical inertial energy is associated with inertance in the acoustical system. Acoustical energy increases as the volume current of an inertance increases. It decreases when the volume current decreases. It remains constant when the volume current is a constant. Inertance is the acoustical element that opposes a change in volume current. Inertance M, in grams per (centimeter)^4, is defined as

$$p = M\frac{dU}{dt} \tag{2.9}$$

where M = inertance, in grams per (centimeter)^4,

dU/dt = rate of change of volume current, in cubic centimeters per second per second, and

p = driving pressure, in dynes per square centimeter.

Equation 2.9 states that the driving pressure applied to an inertance is proportional to the inertance and the rate of change of volume current.

Inertance [3] may be expressed as

$$M = \frac{m}{S^2} \tag{2.10}$$

where m = mass, in grams,

S = cross sectional area in square centimeters, over which the driving pressure acts to drive the mass.

The inertance of a circular tube is

$$M = \frac{\rho l}{\pi R^2} \tag{2.11}$$

where R = radius of the tube, in centimeters,

l = effective length of the tube, that is, length plus end correction, in centimeters, and

ρ = density of the medium in the tube, in grams per cubic centimeter.

[3] Olson, "Acoustical Engineering," D. Van Nostrand Co., Princeton, N. J., 1957.

2.4. Electrical Capacitance, Rectilineal Compliance, Rotational Compliance, Acoustical Capacitance

A. *Electrical Capacitance.*—Electrostatic energy is associated with the separation of positive and negative charges as in the case of the charges on the two plates of an electrical capacitance. Electrostatic energy increases as the charges of opposite polarity are separated. It is constant and stored when the charges remain unchanged. It decreases as the charges are brought together and the electrostatic energy released. Electrical capacitance is the electrical circuit element which opposes a change in voltage. Electrical capacitance C_E, in abfarads, is defined as

$$i = C_E \frac{de}{dt} \tag{2.12}$$

Equation 2.12 may be written

$$e = \frac{1}{C_E} \int i dt = \frac{q}{C_E} \tag{2.13}$$

where q = charge on electrical capacitance, in abcoulombs, and
e = electromotive force, in abvolts.

Equation 2.13 states that the charge on an electrical capacitance is proportional to the electrical capacitance and the applied electromotive force.

B. *Rectilineal Compliance.*—Mechanical rectilineal potential energy is associated with the compression of a spring or compliant element. Mechanical energy increases as the spring is compressed. It decreases as the spring is allowed to expand. It is a constant, and is stored, when the spring remains immovably compressed. Rectilineal compliance is the mechanical element which opposes a change in the applied force. Rectilineal compliance C_M (termed compliance) in centimeters per dyne, is defined as

$$f_M = \frac{x}{C_M} \tag{2.14}$$

where x = displacement, in centimeters, and
f_M = applied force, in dynes

Equation 2.14 states that the linear displacement of a compliance is proportional to the compliance and the applied force.

Stiffness is the reciprocal of compliance.

C. *Rotational Compliance.*—Mechanical rotational potential energy is associated with the twisting of a spring or compliant element. Mechanical energy increases as the spring is twisted. It decreases as the spring is allowed to unwind. It is constant, and is stored when the spring remains immovably twisted. Rotational compliance is the mechanical element which opposes a change in the applied torque. Rotational compliance C_R, in radians per centimeter per dyne, is defined as

$$f_R = \frac{\phi}{C_R} \tag{2.15}$$

where ϕ = angular displacement, in radians, and
f_R = applied torque, in dyne centimeters.

Equation 2.15 states that the rotational displacement of the rotational compliance is proportional to the rotational compliance and the applied force.

D. *Acoustical Capacitance.*—Acoustical potential energy is associated with the compression of a fluid or gas. Acoustical energy increases as the gas is compressed. It decreases as the gas is allowed to expand. It is constant, and is stored when the gas remains immovably compressed. Acoustical capacitance is the acoustic element which opposes a change in the applied pressure. The pressure,[4] in dynes per square centimeter, in terms of the condensation, is

$$p = c^2 \rho s \tag{2.16}$$

where c = velocity, in centimeters per second,
ρ = density, in grams per cubic centimeter, and
s = condensation, defined in equation 2.17.

The condensation in a volume V due to a change in volume from V to V' is

$$s = \frac{V - V'}{V} \tag{2.17}$$

[4] Olson, "Acoustical Engineering," p. 10, D. Van Nostrand Co., Princeton, N. J., 1957.

The change in volume $V - V'$, in cubic centimeters, is equal to the volume displacement, in cubic centimeters.

$$V - V' = X \tag{2.18}$$

where X = volume displacement, in cubic centimeters.

From equations 2.16, 2.17, and 2.18 the pressure is

$$p = \frac{\rho c^2}{V} X \tag{2.19}$$

Acoustical capacitance C_A is defined as

$$p = \frac{X}{C_A} \tag{2.20}$$

where p = sound pressure in dynes per square centimeter, and
X = volume displacement, in cubic centimeters.

Equation 2.20 states the volume displacement in an acoustical capacitance is proportional to the pressure and the acoustical capacitance.

From equations 2.19 and 2.20 the acoustical capacitance of a volume is

$$C_A = \frac{V}{\rho c^2} \tag{2.21}$$

where V = volume, in cubic centimeters.

2.5. Representation of Electrical, Mechanical Rectilineal, Mechanical Rotational and Acoustical Elements

Electrical, mechanical rectilineal, mechanical rotational and acoustical elements have been defined in the preceding sections. Fig. 2.1 illustrates schematically the three elements in each of the four systems.

The electrical elements, electrical resistance, inductance and electrical capacitance are represented by the conventional symbols.

Mechanical rectilineal resistance is represented by sliding friction which causes dissipation. Mechanical rotational resistance is represented by a wheelwith a sliding friction brake which causes dissipation. Acoustical resistance is represented by narrow slits which causes dissipation due to viscosity when fluid is forced through the slits. These elements are analogous to electrical resistance in the electrical system.

Inertia in the mechanical rectilineal system is represented by a mass. Moment of inertia in the mechanical rotational system is represented by a flywheel. Inertance in the acoustical system is represented as the fluid contained in a tube in which all the particles move with the same phase when actuated by a force due to pressure. These elements are analogous to inductance in the electrical system.

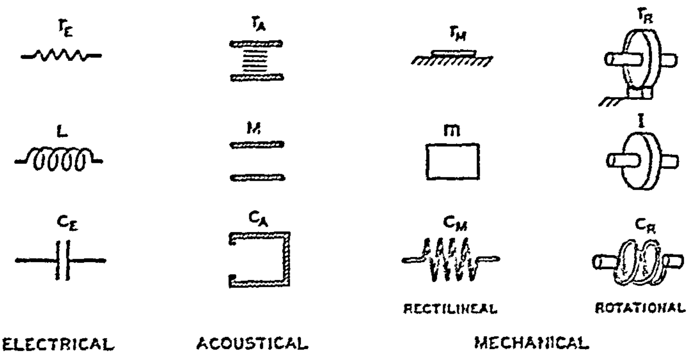

FIG. 2.1. Graphical representation of the three basic elements in electrical, mechanical rectilineal, mechanical rotational and acoustical systems.

r_E = electrical resistance	r_A = acoustical resistance	r_M = mechanical rectilineal resistance	r_R = mechanical rotational resistance
L = inductance	M = inertance	m = mass	I = moment of inertia
C_E = electrical capacitance	C_A = acoustical capacitance	C_M = compliance	C_R = rotational compliance

Compliance in the mechanical rectilineal system is represented as a spring. Rotational compliance in the mechanical rotational system is represented as a spring. Acoustical capacitance in the acoustical system is represented as a volume which acts as a stiffness or spring element. These elements are analogous to electrical capacitance in the electrical system.

In the preceding discussion of electrical, mechanical rectilineal, mechanical rotational and acoustical systems it was observed that the four systems are analogous. As pointed out in the introduction, using the dynamical concept for flow of electrical currents in electrical circuits the fundamental laws are of the same nature as those which govern the dynamics of a moving body. In general, the three fundamental dimen-

TABLE 2.1

Electrical		Mechanical Rectilineal		Mechanical Rotational		Acoustical	
Quantity	Symbol	Quantity	Symbol	Quantity	Symbol	Quantity	Symbol
Self-Inductance	L	Mass	m	Moment of Inertia	I	Inertance	M
Electrical Charge	q	Linear Displacement	x	Angular Displacement	ϕ	Volume Displacement	X
Time	t	Time	t	Time	t	Time	t

TABLE 2.2

Electrical			Mechanical Rectilineal			Mechanical Rotational			Acoustical		
Quantity	Symbol	Dimension	Quantity	Symbol	Dimension	Quantity	Symbol	Dimension	Quantity	Symbol	Dimension
Current	i	qt^{-1}	Linear Velocity	$\dot{x}$ or v	xt^{-1}	Angular Velocity	$\dot{\phi}$ or θ	ϕt^{-1}	Volume Current	$\dot{X}$ or U	Xt^{-1}
Electromotive Force	e	Lqt^{-2}	Force	f_M	mxt^{-2}	Torque	f_R	$I\phi t^{-2}$	Pressure	p	MXt^{-2}
Electrical Resistance	r_E	Lt^{-1}	Mechanical Resistance	r_M	mt^{-1}	Rotational Resistance	r_R	It^{-1}	Acoustical Resistance	r_A	Mt^{-1}
Electrical Capacitance	C_E	$L^{-1}t^2$	Compliance	C_M	$m^{-1}t^2$	Rotational Compliance	C_R	$I^{-1}t^2$	Acoustical Capacitance	C_A	$M^{-1}t^2$
Energy	W_E	Lq^2t^{-2}	Energy	W_M	mx^2t^{-2}	Energy	W_R	$I\phi^2t^{-2}$	Energy	W_A	MX^2t^{-2}
Power	P_E	Lq^2t^{-3}	Power	P_M	mx^2t^{-3}	Power	P_R	$I\phi^2t^{-3}$	Power	P_A	MX^2t^{-3}

TABLE 2.3

Electrical				Mechanical Rectilineal			
Quantity	Unit	Symbol	Dimension	Quantity	Unit	Symbol	Dimension
Electromotive Force	Volts $\times 10^8$	e	$M^{1/2}L^{3/2}T^{-2}$	Force	Dynes	f_M	MLT^{-2}
Charge or Quantity	Coulombs $\times 10^{-1}$	q	$M^{1/2}L^{1/2}$	Linear Displacement	Centimeters	x	L
Current	Amperes $\times 10^{-1}$	i	$M^{1/2}L^{1/2}T^{-1}$	Linear Velocity	Centimeters per Second	$\dot{x}$ or v	LT^{-1}
Electrical Impedance	Ohms $\times 10^9$	z_E	LT^{-1}	Mechanical Impedance	Mechanical Ohms	z_M	MT^{-1}
Electrical Resistance	Ohms $\times 10^9$	r_E	LT^{-1}	Mechanical Resistance	Mechanical Ohms	r_M	MT^{-1}
Electrical Reactance	Ohms $\times 10^9$	x_E	LT^{-1}	Mechanical Reactance	Mechanical Ohms	x_M	MT^{-1}
Inductance	Henries $\times 10^9$	L	L	Mass	Grams	m	M
Electrical Capacitance	Farads $\times 10^{-9}$	C_E	$L^{-1}T^2$	Compliance	Centimeters per Dyne	C_M	$M^{-1}T^2$
Power	Ergs per Second	P_E	ML^2T^{-3}	Power	Ergs per Second	P_M	ML^2T^{-3}

TABLE 2.3—*Continued*

Mechanical Rotational				Acoustical			
Quantity	Unit	Symbol	Dimension	Quantity	Unit	Symbol	Dimension
Torque	Dyne Centimeter	f_R	ML^2T^{-2}	Pressure	Dynes per Square Centimeter	p	$ML^{-1}T^{-2}$
Angular Displacement	Radians	ϕ	1	Volume Displacement	Cubic Centimeters	X	L^3
Angular Velocity	Radians per Second	$\dot{\phi}$ or θ	T^{-1}	Volume Current	Cubic Centimeters per Second	$\dot{X}$ or U	L^3T^{-1}
Rotational Impedance	Rotational Ohms	z_R	ML^2T^{-1}	Acoustical Impedance	Acoustical Ohms	z_A	$ML^{-4}T^{-1}$
Rotational Resistance	Rotational Ohms	r_R	ML^2T^{-1}	Acoustical Resistance	Acoustical Ohms	r_A	$ML^{-4}T^{-1}$
Rotational Reactance	Rotational Ohms	x_R	ML^2T^{-1}	Acoustical Reactance	Acoustical Ohms	x_A	$ML^{-4}T^{-1}$
Moment of Inertia	(Gram) (Centimeter)^2	I	ML^2	Inertance	Grams per (Centimeter)^4	M	ML^{-4}
Rotational Compliance	Radians per Dyne per Centimeter	C_R	$M^{-1}L^{-2}T^2$	Acoustical Capacitance	(Centimeter)^5 per Dyne	C_A	$M^{-1}L^4T^2$
Power	Ergs per Second	P_R	ML^2T^{-3}	Power	Ergs per Second	P_A	ML^2T^{-3}

CHAPTER III

ELECTRICAL, MECHANICAL RECTILINEAL, MECHANICAL ROTATIONAL, AND ACOUSTICAL SYSTEMS OF ONE DEGREE OF FREEDOM

3.1. Introduction

In the preceding sections the fundamental elements in each of the four systems have been defined. From these definitions it is evident that friction, mass, and compliance govern the movements of physical bodies in the same manner that resistance, inductance and capacitance govern the movement of electricity. In any dynamical system there are two distinct problems; namely, the derivation of the differential equation from the statement of the problem and the physical laws, and the solution of the differential equation. It is the purpose of this chapter to establish and solve the differential equations for electrical, mechanical rectilineal, mechanical rotational and acoustical systems of one degree of freedom. These equations will show that the coefficients in the differential equation of the electrical system are elements in the electrical circuit. In the same way the coefficients in the differential equations of the mechanical rectilineal, mechanical rotational and acoustical systems may be looked upon as mechanical rectilineal, mechanical rotational or acoustical elements. In other words, a consideration of the four systems of a single degree of freedom provides another means of establishing the analogies between electrical, mechanical rectilineal, mechanical rotational and acoustical systems.

3.2. Description of Systems of One Degree of Freedom

An electrical, mechanical rectilineal, mechanical rotational, and acoustical system of one degree of freedom is shown in Fig. 3.1. In one degree of freedom the activity in every element of the system may be expressed in terms of one variable. In the electrical system an electromotive force e acts upon an inductance L, an electrical resistance r_E and an electrical

capacitance C_E connected in series. In the mechanical rectilineal system a driving force f_M acts upon a particle of mass m fastened to a spring or compliance C_M and sliding upon a plate with a frictional force which is proportional to the velocity and designated as the mechanical rectilineal resistance r_M. In the mechanical rotational system a driving torque f_R acts upon a flywheel of moment of inertia I connected to a spring or rotational compliance C_R and the periphery of the wheel sliding against a brake with a frictional force which is proportional to the velocity and designated as the mechanical rotational resistance r_R. In

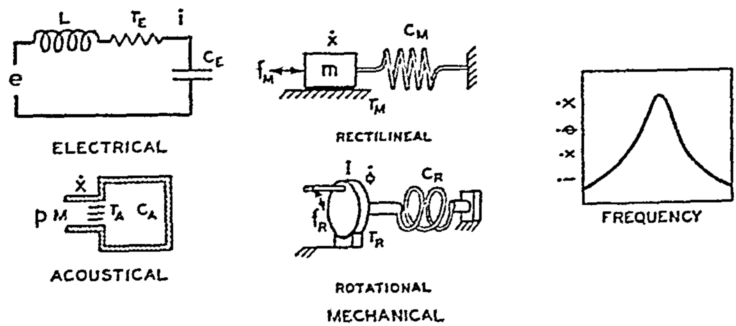

FIG. 3.1. Electrical, mechanical rectilineal, mechanical rotational and acoustical systems of one degree of freedom and the current, velocity, angular velocity and volume current response characteristics.

the acoustical system an impinging sound wave of pressure p acts upon an inertance M and an acoustical resistance r_A comprising the air in the tubular opening which is connected to the volume or acoustical capacitance C_A. The acoustical resistance r_A is due to viscosity.

The principle of the conservation of energy forms one of the basic theorems in most sciences. The principle of conservation of energy states that the total store of energy of all forms remains a constant if the system is isolated so that it neither receives nor gives out energy; in case of transfer of energy the total gain or loss from the system is equal to the loss or gain outside the system. In the electrical, mechanical rectilineal, mechanical rotational, and acoustical systems energy will be confined to three forms; namely, kinetic, potential and heat energy. Kinetic energy of a system is that possessed by virtue of its velocity. Potential energy of a system is that possessed by virtue of its configuration or deformation. Heat is a transient form of energy. In the four

systems; electrical, mechanical rectilineal, mechanical rotational, and acoustical energy is transformed into heat in the dissipative part of the system. The heat energy is carried away either by conduction or radiation. The sum of the kinetic, potential, and heat energy during an interval of time is, by the principle of conservation of energy, equal to the energy delivered to the system during that interval.

3.3. Kinetic Energy

The kinetic energy T_{KE} stored in the magnetic field of the electrical circuit is

$$T_{KE} = \tfrac{1}{2}Li^2 \tag{3.1}$$

where L = inductance, in abhenries, and
i = current through the inductance L, in abamperes.

The kinetic energy T_{KM} stored in the mass of the mechanical rectilineal system is

$$T_{KM} = \tfrac{1}{2}m\dot{x}^2 \tag{3.2}$$

where m = mass, in grams, and
$\dot{x}$ = velocity of the mass m, in centimeters per second.

The kinetic energy T_{KR} stored in the moment of inertia of the mechanical rotational system is

$$T_{KR} = \tfrac{1}{2}I\dot{\phi}^2 \tag{3.3}$$

where I = moment of inertia, in gram (centimeter)^2 and
$\dot{\phi}$ = angular velocity of I, in radians per second.

The kinetic energy T_{KA} stored in the inertance of the acoustical system is

$$T_{KA} = \tfrac{1}{2}M\dot{X}^2 \tag{3.4}$$

where $M = m/S^2$, the inertance, in grams per (centimeter)^4,
m = mass of air in the opening, in grams,
S = cross-sectional area of the opening, in square centimeters,
$\dot{X} = S\dot{x}$ = volume current, in cubic centimeters per second,
$\dot{x}$ = velocity of the air particles in the opening, in centimeters per second.

It is assumed that all the air particles in the opening move with the same phase.

3.4. Potential Energy

The potential energy V_{PE} stored in the electrical capacitance of the electrical circuit is

$$V_{PE} = \frac{1}{2}\frac{q^2}{C_E} \qquad 3.5$$

where C_E = capacitance, in abfarads, and
q = charge on the capacitance, in abcoulombs.

The potential energy V_{PM} stored in the compliance or spring of the mechanical rectilineal system is

$$V_{PM} = \frac{1}{2}\frac{x^2}{C_M} \qquad 3.6$$

where $C_M = 1/s$ = compliance of the spring, in centimeters per dyne,
s = stiffness of the spring, in dynes per centimeter, and
x = displacement, in centimeters.

The potential energy V_{PR} stored in the rotational compliance or spring of the mechanical rotational system is

$$V_{PR} = \frac{1}{2}\frac{\phi^2}{C_R} \qquad 3.7$$

where C_R = rotational compliance of the spring, in radians per dyne per centimeter, and
ϕ = angular displacement, in radians.

The potential energy V_{PA} stored in the acoustical capacitance of the acoustical system is

$$V_{PA} = \frac{1}{2}\frac{X^2}{C_A} \qquad 3.8$$

where X = volume displacement, in cubic centimeters,
$C_A = V/\rho c^2$ = acoustical capacitance, in (centimeters)^5 per dyne,
V = volume of the cavity, in cubic centimeters,
ρ = density of air, in grams per cubic centimeter, and
c = velocity of sound, in centimeters per second.

The energies stored in the systems is the sum of the kinetic and potential energy. The total energy stored in the four systems may be written

$$W_E = T_{KE} + V_{PE} = \tfrac{1}{2}Li^2 + \frac{1}{2}\frac{q^2}{C_E} \qquad 3.9$$

$$W_M = T_{KM} + V_{PM} = \tfrac{1}{2}m\dot{x}^2 + \frac{1}{2}\frac{x^2}{C_M} \qquad 3.10$$

$$W_R = T_{KR} + V_{PR} = \tfrac{1}{2}I\dot{\phi}^2 + \frac{1}{2}\frac{\phi^2}{C_R} \qquad 3.11$$

$$W_A = T_{KA} + V_{PA} = \tfrac{1}{2}M\dot{X}^2 + \frac{1}{2}\frac{X^2}{C_A} \qquad 3.12$$

where W_E, W_M, W_R, and W_A are the total energies stored in electrical, mechanical rectilineal, mechanical rotational, and acoustical systems.

The rate of change of energy with respect to time in the four systems may be written

$$\frac{dW_E}{dt} = Li\frac{di}{dt} + \frac{q\dot{q}}{C_E} = L\dot{q}\ddot{q} + \frac{q\dot{q}}{C_E} \qquad 3.13$$

$$\frac{dW_M}{dt} = m\dot{x}\ddot{x} + \frac{x\dot{x}}{C_M} \qquad 3.14$$

$$\frac{dW_R}{dt} = I\dot{\phi}\ddot{\phi} + \frac{\phi\dot{\phi}}{C_R} \qquad 3.15$$

$$\frac{dW_A}{dt} = M\dot{X}\ddot{X} + \frac{X\dot{X}}{C_A} \qquad 3.16$$

3.5. Dissipation

The rate at which electromagnetic energy D_E is converted into heat is

$$D_E = r_E i^2 \qquad 3.17$$

where r_A = electrical resistance, in abohms, and
i = current, in abamperes.

Assume that the frictional force f_M upon the mass m as it slides back and forth is proportional to the velocity as follows:

$$f_M = r_M\dot{x} \qquad 3.18$$

where r_M = mechanical resistance, in mechanical ohms, and
$\dot{x}$ = velocity, in centimeters per second.

The rate at which mechanical rectilineal energy D_M is converted into heat is

$$D_M = f_M\dot{x} = r_M\dot{x}^2 \qquad 3.19$$

Assume that the frictional torque f_R upon the flywheel I as the periphery of the wheel slides against the brake is proportional to the velocity as follows:

$$f_R = r_R\dot{\phi} \qquad 3.20$$

where r_R = mechanical rotational resistance, in rotational ohms, and
$\dot{\phi}$ = angular velocity, in radians per second.

The rate at which mechanical rotational energy D_R is converted into heat is

$$D_R = f_R\dot{\phi} = r_R\dot{\phi}^2 \qquad 3.21$$

The acoustical energy is converted into heat by the dissipation due to viscosity as the fluid is forced through the narrow slits. The rate at which acoustical energy D_A is converted into heat is

$$D_A = r_A\dot{X}^2 \qquad 3.22$$

where r_A = acoustical resistance, in acoustical ohms, and
$\dot{X}$ = volume current in cubic centimeters per second.

3.6. Equations of Motion

The power delivered to a system must be equal to the rate of kinetic energy storage plus the rate of potential energy storage plus the power loss due to dissipation. The rate at which work is done or power delivered to the electrical system by the applied electromotive force is $\dot{q}E\epsilon^{j\omega t} = e\dot{q}$. The rate at which work is done or power delivered to the mechanical rectilineal system by the applied mechanical force is $\dot{x}F_M\epsilon^{j\omega t} = f_M\dot{x}$. The rate at which work is done or power delivered to the mechanical rotational system by the applied mechanical torque is $\dot{\phi}F_R\epsilon^{j\omega t} = f_R\dot{\phi}$. The rate at which work is done or power delivered to the acoustical system by the applied sound pressure is $\dot{X}P\epsilon^{j\omega t} = p\dot{X}$.

The rate of decrease of energy $(T_K + V_P)$ of the system plus the rate at which work is done on the system or power delivered to the system

by the external forces must equal the rate of dissipation of energy. Writing this sentence mathematically yields the equations of motion for the four systems.

Electrical

$$L\dot{q}\ddot{q} + r_E\dot{q}^2 + \frac{q\dot{q}}{C_E} = E\epsilon^{j\omega t}\dot{q} \qquad 3.23$$

$$L\ddot{q} + r_E\dot{q} + \frac{q}{C_E} = E\epsilon^{j\omega t} \qquad 3.24$$

Mechanical Rectilineal

$$m\dot{x}\ddot{x} + r_M\dot{x}^2 + \frac{x\dot{x}}{C_M} = F_M\epsilon^{j\omega t}\dot{x} \qquad 3.25$$

$$m\ddot{x} + r_M\dot{x} + \frac{x}{C_M} = F_M\epsilon^{j\omega t} \qquad 3.26$$

Mechanical Rotational

$$I\dot{\phi}\ddot{\phi} + r_R\dot{\phi}^2 + \frac{\phi\dot{\phi}}{C_R} = F_R\epsilon^{j\omega t}\dot{\phi} \qquad 3.27$$

$$I\ddot{\phi} + r_R\dot{\phi} + \frac{\phi}{C_R} = F_R\epsilon^{j\omega t} \qquad 3.28$$

Acoustical

$$M\dot{X}\ddot{X} + r_A\dot{X}^2 + \frac{X\dot{X}}{C_A} = P\epsilon^{j\omega t}\dot{X} \qquad 3.29$$

$$M\ddot{X} + r_A\dot{X} + \frac{X}{C_A} = P\epsilon^{j\omega t} \qquad 3.30$$

The steady state solutions of the four differential equations 3.24, 3.26, 3.28 and 3.30 are

Electrical

$$\dot{q} = i = \frac{E\epsilon^{j\omega t}}{r_E + j\omega L - \dfrac{j}{\omega C_E}} = \frac{e}{z_E} \qquad 3.31$$

Mechanical Rectilineal

$$\dot{x} = \frac{F\epsilon^{j\omega t}}{r_M + j\omega m - \dfrac{j}{\omega C_M}} = \frac{f_M}{z_M} \qquad 3.32$$

Mechanical Rotational

$$\dot{\phi} = \frac{F\epsilon^{j\omega t}}{r_R + j\omega I - \dfrac{j}{\omega C_R}} = \frac{f_R}{z_R} \tag{3.33}$$

Acoustical

$$\dot{X} = \frac{p\epsilon^{j\omega t}}{r_A + j\omega M - \dfrac{j}{\omega C_A}} = \frac{p}{z_A} \tag{3.34}$$

The vector electrical impedance is

$$z_E = r_E + j\omega L - \frac{j}{\omega C_E} \tag{3.35}$$

The vector mechanical rectilineal impedance is

$$z_M = r_R + j\omega m - \frac{j}{\omega C_M} \tag{3.36}$$

The vector mechanical rotational impedance is

$$z_R = r_R + j\omega I - \frac{j}{\omega C_R} \tag{3.37}$$

The vector acoustical impedance is

$$z_A = r_A + j\omega M - \frac{j}{\omega C_A} \tag{3.38}$$

3.7. Resonant Frequency

For a certain value of L and C_E, m and C_M, I and C_R, and M and C_A there will be a certain frequency at which the imaginary component of the impedance is zero. This frequency is called the resonant frequency. At this frequency the ratio of the current to the applied voltage or the ratio of the velocity to the applied force or the ratio of the angular velocity to the applied torque or the ratio of the volume current to the applied pressure is a maximum. At the resonant frequency the current and voltage, the velocity and force, the angular velocity and torque, and the volume current and pressure are in phase.

The resonant frequency f_r in the four systems is

Electrical

$$f_r = \frac{1}{2\pi\sqrt{LC_E}} \tag{3.39}$$

Mechanical Rectilineal

$$f_r = \frac{1}{2\pi\sqrt{mC_M}} \tag{3.40}$$

Mechanical Rotational

$$f_r = \frac{1}{2\pi\sqrt{IC_R}} \tag{3.41}$$

Acoustical

$$f_r = \frac{1}{2\pi\sqrt{MC_A}} \tag{3.42}$$

3.8. Kirchhoff's Law and D'Alembert's Principle [1]

Kirchhoff's electromotive force law plays the same role in setting up the electrical equations as D'Alembert's principle does in setting up mechanical and acoustical equations. It is the purpose of this section to obtain the differential equations of electrical, mechanical rectilineal, mechanical rotational and acoustical systems employing Kirchhoff's law and D'Alembert's principle.

Kirchhoff's law is as follows: The algebraic sum of the electromotive forces around a closed circuit is zero. The differential equations for electric circuits with lumped elements may be set up employing Kirchhoff's law. The electromotive forces due to the elements in an electric circuit are

$$\text{Electromotive force of self-inductance} = -L\frac{di}{dt} = -L\frac{d^2q}{dt^2} \tag{3.43}$$

$$\text{Electromotive force of electrical resistance} = -r_E i = -r_E\frac{dq}{dt} \tag{3.44}$$

$$\text{Electromotive force of electrical capacitance} = -\frac{q}{C_E} \tag{3.45}$$

In addition to the above electromotive forces are the electromotive forces applied externally.

[1] D'Alembert's principle as used here may be said to be a modified form of Newton's second law.

The above law may be used to derive the differential equation for the electrical circuit of Fig. 3.1. From Kirchhoff's law the algebraic sum of the electromotive forces around the circuit is zero. The equation may be written

$$L\frac{di}{dt} + r_E i + \frac{q}{C_E} = E\epsilon^{j\omega t} \tag{3.46}$$

where $e = E\epsilon^{j\omega t}$ = the external applied electromotive force.

Equation 3.46 may be written

$$L\frac{d^2q}{dt^2} + r_E\frac{dq}{dt} + \frac{q}{C_E} = E\epsilon^{j\omega t} \tag{3.47}$$

and is the same as equation 3.24.

The differential equations for mechanical systems may be set up employing D'Alembert's principle; namely, the algebraic sum of the forces applied to a body is zero.

The mechanical forces due to the elements in a mechanical rectilineal system are

Mechanomotive force of mass reaction $= -m\dfrac{d^2x}{dt^2}$ 3.48

Mechanomotive force of mechanical rectilineal resistance $= -r_M\dfrac{dx}{dt}$ 3.49

Mechanomotive force of mechanical compliance $= -\dfrac{x}{C_M}$ 3.50

In addition to the above mechanomotive forces are the mechanomotive forces applied externally.

The above principle may be used to derive the differential equation of the mechanical rectilineal system of Fig. 3.1. From D'Alembert's principle the algebraic sum of the forces applied to a body is zero. The equation may be written

$$m\frac{d^2x}{dt^2} + r_M\frac{dx}{dt} + \frac{x}{C_M} = F_M\epsilon^{j\omega t} \tag{3.51}$$

where $f_M = F_M\epsilon^{j\omega t}$ = external applied mechanical force.

Equation 3.51 is the same as equation 3.26.

D'Alembert's principle may be applied to the mechanical rotational system. The rotational mechanical forces due to the elements in a mechanical rotational system are

$$\text{Rotatomotive force of moment of inertia reaction} = -I\frac{d^2\phi}{dt^2} \tag{3.52}$$

Rotatomotive force of mechanical rotational resistance =

$$-r_R\frac{d\phi}{dt} \tag{3.53}$$

$$\text{Rotatomotive force of rotational compliance} = -\frac{\phi}{C_R} \tag{3.54}$$

In addition to the above rotatomotive forces are the rotatomotive forces applied externally.

Applying D'Alembert's principle the equation for the rotational system of Fig. 3.1 may be written

$$I\frac{d^2\phi}{dt^2} + r_R\frac{d\phi}{dt} + \frac{\phi}{C_R} = F_R\epsilon^{j\omega t} \tag{3.55}$$

where $f_R = F_R\epsilon^{j\omega t}$ = external applied torque.

Equation 3.55 is the same as equation 3.28.

D'Alembert's principle may be applied to the acoustical system. The acoustical pressures due to the elements in an acoustical system are

$$\text{Acoustomotive force of inertive reaction} = -M\frac{d^2X}{dt^2} \tag{3.56}$$

$$\text{Acoustomotive force of acoustical resistance} = -r_A\frac{dx}{dt} \tag{3.57}$$

$$\text{Acoustomotive force of acoustical capacitance} = -\frac{X}{C_A} \tag{3.58}$$

In addition to the above acoustomotive forces are the acoustomotive forces applied externally.

Applying D'Alembert's principle, the equation for the acoustical system of Fig. 3.1 may be written

$$M\frac{d^2X}{dt^2} + r_A\frac{dX}{dt} + \frac{X}{C_A} = P\epsilon^{j\omega t} \qquad 3.59$$

where $p = P\epsilon^{j\omega t}$ = external applied pressure.

Equation 3.59 is the same as equation 3.30.

Equations 3.43 to 3.59, inclusively, further illustrate the analogies between electrical, mechanical rectilineal, mechanical rotational, and acoustical systems.

CHAPTER IV

ELECTRICAL, MECHANICAL RECTILINEAL, MECHANICAL ROTATIONAL AND ACOUSTICAL SYSTEMS OF TWO AND THREE DEGREES OF FREEDOM

4.1. Introduction

The analogies between the four types of vibrating systems of one degree of freedom have been considered in the preceding chapter. It is the purpose of this section to extend these analogies to systems of two and three degrees of freedom. In this chapter the differential equations for the four systems will be obtained from the expressions for the kinetic and potential energies, the dissipation and the application of Lagrange's equations.

4.2. Two Degrees of Freedom

The first consideration will be the systems shown in Fig. 4.1. In the electrical system an electromotive force acts upon an electrical capaci-

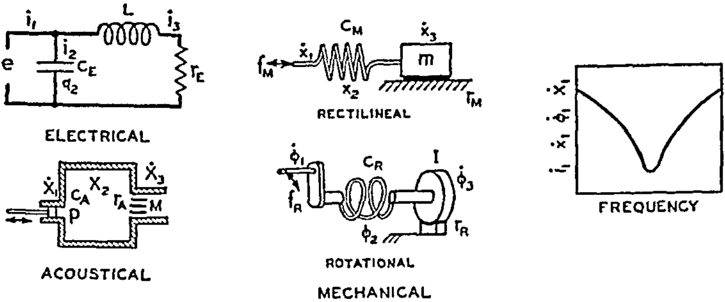

FIG. 4.1. Electrical, mechanical rectilineal, mechanical rotational and acoustical systems of two degrees of freedom and the input current, velocity, angular velocity and volume current response characteristics.

tance C_E shunted by an inductance L and an electrical resistance r_E in series. In the mechanical rectilineal system a driving force acts upon a

spring or compliance C_M connected to a mass m sliding upon a plate with a frictional force which is proportional to the velocity and designated as the mechanical rectilineal resistance r_M. In the mechanical rotational system a driving torque acts upon a spring or rotational compliance C_R connected to a flywheel of moment of inertia I and with the periphery of the wheel sliding against a brake with a frictional force which is proportional to the velocity and designated as the mechanical rotational resistance r_R. In the acoustical system a driving pressure p acts upon a volume or acoustical capacitance C_A connected to a tubular opening communicating with free space. The mass of fluid in the opening is the inertance M and the fluid resistance produced by the slits is the acoustical resistance r_A.

4.3. Kinetic Energy

The kinetic energy T_{KE} stored in the magnetic field of the electrical circuit is

$$T_{KE} = \tfrac{1}{2}L\dot{q}_3^2 \tag{4.1}$$

where L = inductance, in abhenries, and
$\dot{q}_3 = i_3$ = current, in branch 3, in abamperes.

The kinetic energy T_{KM} stored in the mass of the mechanical rectilineal system is

$$T_{KM} = \tfrac{1}{2}m\dot{x}_3^2 \tag{4.2}$$

where m = mass, in grams, and
$\dot{x}_3$ = velocity of the mass m, in centimeters per second.

The kinetic energy T_{KR} stored in the moment of inertia of the mechanical rotational system is

$$T_{KR} = \tfrac{1}{2}I\dot{\phi}_3^2 \tag{4.3}$$

where I = moment of inertia, in gram (centimeter)^2 and
$\dot{\phi}_3$ = angular velocity of I, in radians per second.

The kinetic energy T_{KA} stored in the inertance of the acoustical system is

$$T_{KA} = \tfrac{1}{2}M\dot{X}_3^2 \tag{4.4}$$

where M = inertance, in grams per (centimeter)^4 and
$\dot{X}_3$ = volume current, in cubic centimeters per second.

4.4. Potential Energy

The potential energy V_{PE} stored in the electric field of the electrical circuit is

$$V_{PE} = \frac{1}{2}\frac{q_2^2}{C_E} \tag{4.5}$$

where C_E = capacitance, in abfarads, and
q_2 = charge on the electrical capacitance, in abcoulombs.

The potential energy V_{PM} stored in the compliance or spring of the mechanical rectilineal system is

$$V_{PM} = \frac{1}{2}\frac{x_2^2}{C_M} \tag{4.6}$$

where C_M = compliance of the spring, in centimeters per dyne, and
x_2 = displacement, in centimeters.

The potential energy V_{PR} stored in the rotational compliance or spring of the mechanical rotational system is

$$V_{PR} = \frac{1}{2}\frac{\phi_2^2}{C_R} \tag{4.7}$$

where C_R = rotational compliance of the spring, in radians per dyne per centimeter, and
ϕ_2 = angular displacement, in radians.

The potential energy V_{PA} stored in the acoustical capacitance of the acoustical system is

$$V_{PA} = \frac{1}{2}\frac{X_2^2}{C_A} \tag{4.8}$$

where C_A = acoustical capacitance, in (centimeter)^5 per dyne, and
X_2 = volume displacement, in cubic centimeters.

4.5. Dissipation

The rate at which electromagnetic energy D_E is converted into heat is

$$D_E = r_E i_3^2 = r_E \dot{q}_3^2 \tag{4.9}$$

where r_E = electrical resistance, in ohms, and
$i_3 = \dot{q}_3$ = current, in abamperes.

The rate at which mechanical rectilineal energy D_M is converted into heat is

$$D_M = r_M \dot{x}_3^2 \tag{4.10}$$

where r_M = mechanical rectilineal resistance, in mechanical ohms, and
$\dot{x}_3$ = velocity, in centimeters per second.

The rate at which mechanical rotational energy D_R is converted into heat is

$$D_R = r_R \dot{\phi}_3^2 \tag{4.11}$$

where r_R = mechanical rotational resistance, in rotational ohms, and
$\dot{\phi}_3$ = angular velocity, in radians per second.

The rate at which acoustical energy D_A is converted into heat is

$$D_A = r_A \dot{X}_3^2 \tag{4.12}$$

where r_A = acoustical resistance, in acoustical ohms, and
$\dot{X}_3$ = volume current, in cubic centimeters per second.

4.6. Equations of Motion

Lagrange's equations for the four systems are as follows:

Electrical

$$\frac{\partial}{\partial t}\left(\frac{\partial T}{\partial \dot{q}_n}\right) - \frac{\partial(T - V)}{\partial q_n} + \frac{1}{2}\frac{\partial D}{\partial \dot{q}_n} = e_n \tag{4.13}$$

where n = number independent coordinates.

Mechanical Rectilineal

$$\frac{\partial}{\partial t}\left(\frac{\partial T}{\partial \dot{x}_n}\right) - \frac{\partial(T - V)}{\partial x_n} + \frac{1}{2}\frac{\partial D}{\partial \dot{x}_n} = f_{Mn} \tag{4.14}$$

Mechanical Rotational

$$\frac{\partial}{\partial t}\left(\frac{\partial T}{\partial \dot{\phi}_n}\right) - \frac{\partial(T - V)}{\partial \phi_n} + \frac{1}{2}\frac{\partial D}{\partial \dot{\phi}_n} = f_{Rn} \tag{4.15}$$

Acoustical

$$\frac{\partial}{\partial t}\left(\frac{\partial T}{\partial \dot{X}_n}\right) - \frac{\partial(T - V)}{\partial X_n} + \frac{1}{2}\frac{\partial D}{\partial \dot{X}_n} = p_n \tag{4.16}$$

4.7. The Electrical System

Applying Lagrange's equation 4.13,

$$e = L\ddot{q}_3 + r_E\dot{q}_3 \qquad 4.17$$

$$e = \frac{q_2}{C_E} \qquad 4.18$$

The electromotive force applied to the inductance and electrical resistance in series is given by equation 4.17. The electromotive force applied to the electrical capacitance in terms of the displacement is given by equation 4.18.

The relation for the currents in Fig. 4.1 is

$$i_1 = i_2 + i_3 \qquad 4.19$$

Equation 4.19 may be written

$$\dot{q}_1 = \dot{q}_2 + \dot{q}_3 \quad \text{or} \qquad 4.20$$

$$q_1 = q_2 + q_3 \qquad 4.21$$

4.8. The Mechanical Rectilineal System

Applying Lagrange's equation 4.14,

$$f_M = m\ddot{x}_3 + r_M\dot{x}_3 \qquad 4.22$$

$$f_M = \frac{x_2}{C_M} \qquad 4.23$$

The force applied to the mass and mechanical rectilineal resistance is given by equation 4.22. The force applied to the spring in terms of the displacement is given by equation 4.23.

The linear displacement, at f_M, in the mechanical rectilineal system of Fig. 4.1 is the sum of the displacement of the mass m and the displacement of the compliance C_M.

$$x_1 = x_2 + x_3 \qquad 4.24$$

Differentiating equation 4.24 with respect to the time the velocities are

$$\dot{x}_1 = \dot{x}_2 + \dot{x}_3 \qquad 4.25$$

4.9. The Mechanical Rotational System

Applying Lagrange's equation 4.15,

$$f_R = I\ddot{\phi}_3 + r_R\dot{\phi}_3 \tag{4.26}$$

$$f_R = \frac{\phi_2}{C_R} \tag{4.27}$$

The torque applied to the flywheel and mechanical rotational resistance is given by equation 4.26. The torque applied to the spring is given by equation 4.27.

The angular displacement, at f_R, in the mechanical rectilineal system of Fig. 4.1 is the sum of the angular displacement of the flywheel I and the angular displacement of the rotational compliance C_R.

$$\phi_1 = \phi_2 + \phi_3 \tag{4.28}$$

Differentiating equation 4.28 with respect to the time the angular velocities are

$$\dot{\phi}_1 = \dot{\phi}_2 + \dot{\phi}_3 \tag{4.29}$$

4.10. The Acoustical System

Applying Lagrange's equation 4.16,

$$p = M\ddot{X}_3 + r_A\dot{X}_3 \tag{4.30}$$

$$p = \frac{X_2}{C_A} \tag{4.31}$$

The pressure applied to the inertance and acoustical resistance is given by equation 4.30. The pressure applied to the acoustical capacitance in terms of the volume displacement is given by equation 4.31.

The volume displacement, at p, in the acoustical system, Fig. 4.1, is the sum of the volume displacement of the inertance M and the volume displacement of the acoustic capacitance C_A.

$$X_1 = X_2 + X_3 \tag{4.32}$$

The volume displacement X_1 is the volume displacement of the vibrating piston. The vibrating piston is not a part of the acoustical system. It is merely the sound pressure source which produces the sound pressure p.

Differentiating equation 4.32 with respect to the time the volume currents are

$$\dot{X}_1 = \dot{X}_2 + \dot{X}_3 \qquad 4.33$$

4.11. Comparison of the Four Systems

A comparison of the coefficients of equations 4.1 to 4.33, inclusive, shows again that resistance, inductance, and capacitance are analogous to mechanical rectilineal resistance, mass, and compliance in the mechanical rectilineal system, to mechanical rotational resistance, moment of inertia and rotational compliance in the mechanical rotational system, and to acoustical resistance, inertance and acoustic capacitance in the acoustical system. A comparison of equations 4.19, 4.25, 4.29 and 4.33 shows that currents in the electrical system are analogous to velocities in the mechanical rectilineal system, to angular velocities in the mechanical rotational system, and to volume currents in the acoustical system.

The current i_3 through the inductance L and electrical resistance r_E, Fig. 4.1, is given by

$$i_3 = \frac{e}{r_E + j\omega L} \qquad 4.34$$

The total current i_1 is given by

$$i_1 = \frac{e\left(r_E + j\omega L + \dfrac{1}{j\omega C_E}\right)}{(r_E + j\omega L)\dfrac{1}{j\omega C_E}} \qquad 4.35$$

The current i_2 through the electrical capacitance C_E is

$$i_2 = i_1 - i_3 = ej\omega C_E \qquad 4.36$$

The linear velocity $\dot{x}_3$ of the mass m and mechanical rectilineal resistance, r_M, Fig. 4.1, is given by

$$\dot{x}_3 = \frac{f_M}{r_M + j\omega m} \qquad 4.37$$

The linear velocity $\dot{x}_1$ at f_M is given by

$$\dot{x}_1 = \frac{f_M\left(r_M + j\omega m + \dfrac{1}{j\omega C_M}\right)}{(r_M + j\omega m)\dfrac{1}{j\omega C_M}} \qquad 4.38$$

The velocity $\dot{x}_2$, the difference in linear velocity between the two ends of the spring C_M, is given by

$$\dot{x}_2 = \dot{x}_1 - \dot{x}_3 = f_M j\omega C_M \tag{4.39}$$

The angular velocity $\dot{\phi}_3$ of the moment of inertia I and mechanical rotational resistance r_R, Fig. 4.1, is given by

$$\dot{\phi}_3 = \frac{f_R}{r_R + j\omega I} \tag{4.40}$$

The total angular velocity $\dot{\phi}_1$ at f_R is given by

$$\dot{\phi}_1 = \frac{f_R\left(r_R + j\omega I + \dfrac{1}{j\omega C_R}\right)}{(r_R + j\omega I)\dfrac{1}{j\omega C_R}} \tag{4.41}$$

The angular velocity $\dot{\phi}_2$, the difference in angular velocity between the two ends of the spring C_R, is given by

$$\dot{\phi}_2 = \dot{\phi}_1 - \dot{\phi}_3 = f_R j\omega C_R \tag{4.42}$$

The volume current $\dot{X}_3$ through the inertance M and the acoustic resistance r_A, Fig. 4.1, is given by

$$\dot{X}_3 = \frac{p}{r_A + j\omega M} \tag{4.43}$$

The total volume current $\dot{X}_1$ at p is given by

$$\dot{X}_1 = \frac{p\left(r_A + j\omega M + \dfrac{1}{j\omega C_A}\right)}{(r_A + j\omega M)\dfrac{1}{j\omega C_A}} \tag{4.44}$$

The volume current $\dot{X}_2$, the difference between the volume currents at the input and output to the acoustical capacitance, is

$$\dot{X}_2 = \dot{X}_1 - \dot{X}_3 = p j\omega C_A \tag{4.45}$$

4.12. Electrical Inductive and Capacitive Coupled Systems of Two Degrees of Freedom and the Mechanical Rectilineal, Mechanical Rotational and Acoustical Analogies

It is the purpose of this section to show two additional electrical arrangements of two degrees of freedom and the mechanical rectilineal,[1] mechanical rotation and acoustical analogies.

The electrical impedances z_{E1}, z_{E2} and z_{E3} in terms of the elements of Fig. 4.2 are as follows:

$$z_{E1} = r_{E1} + j\omega L_1 + \frac{1}{j\omega C_{E1}} \tag{4.46}$$

$$z_{E2} = \frac{1}{j\omega C_{E2}} \tag{4.47}$$

$$z_{E3} = r_{E2} + j\omega L_2 + \frac{1}{j\omega C_{E3}} \tag{4.48}$$

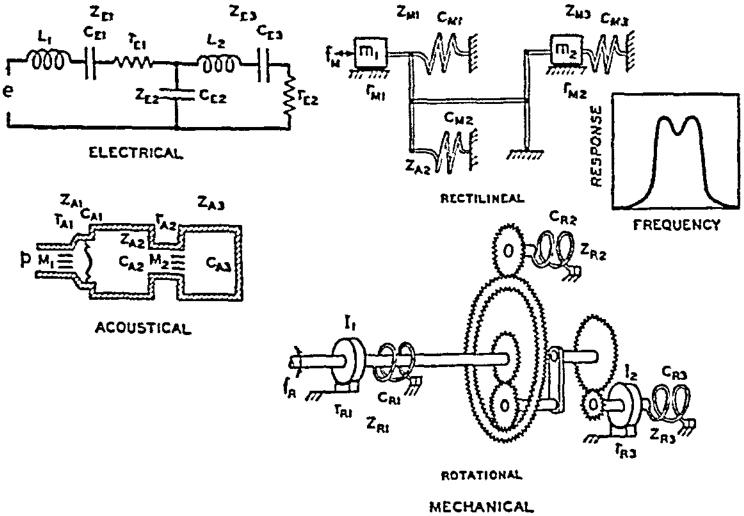

FIG. 4.2. A capacitive coupled electrical system of two degrees of freedom and the mechanical rectilineal, mechanical rotational and acoustical analogies. The graph depicts the output response frequency characteristic.

[1] For an explanation of the shunt mechanical rectilineal and mechanical rotational systems of Figs. 4.2 and 4.3, see pages 59, 60, 61 and 62 and Fig. 5.1 of Chapter V.

The mechanical rectilineal impedances z_{M1}, z_{M2} and z_{M3} in terms of the elements of Fig. 4.2 are as follows:

$$z_{M1} = r_{M1} + j\omega m_1 + \frac{1}{j\omega C_{M1}} \tag{4.49}$$

$$z_{M2} = \frac{1}{j\omega C_{M2}} \tag{4.50}$$

$$z_{M3} = r_{M2} + j\omega m_2 + \frac{1}{j\omega C_{M3}} \tag{4.51}$$

The mechanical rotational impedances z_{R1}, z_{R2} and z_{R3} in terms of the elements of Fig. 4.2 are as follows:

$$z_{R1} = r_{M1} + j\omega I_1 + \frac{1}{j\omega C_{R1}} \tag{4.52}$$

$$z_{R2} = \frac{1}{j\omega C_{R2}} \tag{4.53}$$

$$z_{R3} = r_{R2} + j\omega I_2 + \frac{1}{j\omega C_{R3}} \tag{4.54}$$

The acoustical impedances z_{A1}, z_{A2}, and z_{A3} in terms of the elements of Fig. 4.2 are as follows:

$$z_{A1} = r_{A1} + j\omega M_1 + \frac{1}{j\omega C_{A1}} \tag{4.55}$$

$$z_{A2} = \frac{1}{j\omega C_{A2}} \tag{4.56}$$

$$z_{A3} = r_{A2} + j\omega M_2 + \frac{1}{j\omega C_{A3}} \tag{4.57}$$

The system in Fig. 4.3 is the same as that of Fig. 4.2 save that the shunt electrical capacitance, compliance, rotational compliance and acoustical capacitance, C_{E2}, C_{M2}, C_{R2} and C_{A2}, are replaced by the shunt inductance, mass, moment of inertia and inertance L_2, m_2, I_2 and M_2.

The shunt electrical, mechanical rectilineal, mechanical rotational and acoustical shunt impedances are

$$z_{E2} = j\omega L_2 \tag{4.58}$$

$$z_{M2} = j\omega m_2 \tag{4.59}$$

$$z_{R2} = j\omega I_2 \tag{4.60}$$

$$z_{A2} = j\omega M_2 \tag{4.61}$$

The current in the branch z_{E1} is

$$\dot{i}_1 = \frac{e(z_{E2} + z_{E3})}{z_{E1}z_{E2} + z_{E1}z_{E3} + z_{E2}z_{E3}} \tag{4.62}$$

The velocity of the mass m_1 is

$$\dot{x}_1 = \frac{f_M(z_{M2} + z_{M3})}{z_{M1}z_{M2} + z_{M1}z_{M3} + z_{M2}z_{M3}} \tag{4.63}$$

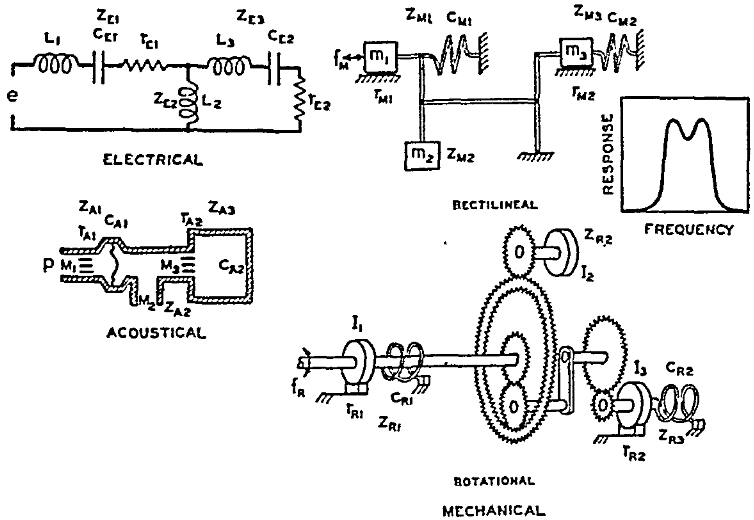

Fig. 4.3. An inductive coupled electrical system of two degrees of freedom and the mechanical rectilineal, mechanical rotational and acoustical analogies. The graph depicts the output response frequency characteristic.

The angular velocity of the moment of inertia I_1 is

$$\dot{\phi}_1 = \frac{f_R(z_{R2} + z_{R3})}{z_{R1}z_{R2} + z_{R1}z_{R3} + z_{R2}z_{R3}} \tag{4.64}$$

The volume current of the inertance M_1 is

$$\dot{X}_1 = \frac{p(z_{A2} + z_{A3})}{z_{A1}z_{A2} + z_{A1}z_{A3} + z_{A2}z_{A3}} \tag{4.65}$$

The current in the branch z_{E3} is

$$i_3 = \frac{e z_{E2}}{z_{E1}z_{E2} + z_{E1}z_{E3} + z_{E2}z_{E3}} \tag{4.66}$$

The velocity of the mass m_3 is

$$\dot{x}_3 = \frac{f_M z_{M2}}{z_{M1}z_{M2} + z_{M1}z_{M3} + z_{M2}z_{M3}} \tag{4.67}$$

The angular velocity of the moment of inertia I_3 is

$$\dot{\phi}_3 = \frac{f_R z_{R2}}{z_{R1}z_{R2} + z_{R1}z_{R3} + z_{R2}z_{R3}} \tag{4.68}$$

The volume current of the inertance M_3 is

$$\dot{X}_3 = \frac{p z_{A2}}{z_{A1}z_{A2} + z_{A1}z_{A3} + z_{A2}z_{A3}} \tag{4.69}$$

The response frequency characteristic of the system is shown in Figs. 4.2 and 4.3.

4.13. Electrical, Mechanical Rectilineal, Mechanical Rotational and Acoustical Systems of Three Degrees of Freedom

Systems of three degrees of freedom are shown in Fig. 4.4. Following the procedures outlined in the preceding sections it can be shown that

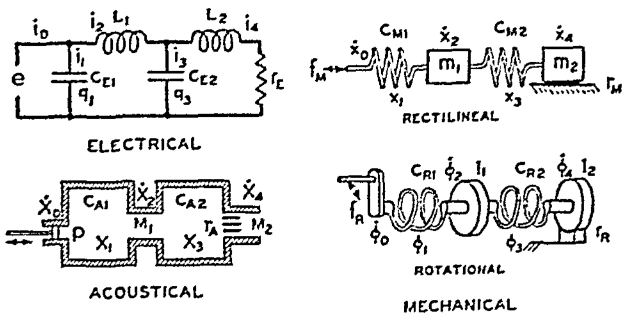

FIG. 4.4. Electrical, mechanical rectilineal, mechanical rotational and acoustical systems of three degrees of freedom.

L_1, L_2, C_{E1}, C_{E2} and r_E in the electrical system are equivalent to m_1, m_2, C_{M1}, C_{M2} and r_M in the mechanical rectilineal system, to I_1, I_2, C_{R1}, C_{R2}

and r_R in the mechanical rotational system and to M_1, M_2, C_{A1}, C_{A2} and r_A in the acoustical system. These equations also show that i_0, i_1, i_2, i_3 and i_4 in the electrical system are equivalent to $\dot{x}_0$, $\dot{x}_1$, $\dot{x}_2$, $\dot{x}_3$ and $\dot{x}_4$ in the mechanical rectilineal system, to $\dot{\phi}_0$, $\dot{\phi}_1$, $\dot{\phi}_2$, $\dot{\phi}_3$ and $\dot{\phi}_4$ in the mechanical rotational system and to $\dot{X}_0$, $\dot{X}_1$, $\dot{X}_2$, $\dot{X}_3$ and to $\dot{X}_4$ in the acoustical system.

The current i_0, the linear velocity $\dot{x}_0$, the angular velocity $\dot{\phi}_0$ and the volume current $\dot{X}_0$ are given by

$$i_0 = \frac{e[(z_{E1} + z_{E2})(z_{E3} + z_{E4}) + z_{E3}z_{E4}]}{H_E} \qquad 4.70$$

$$\dot{x}_0 = \frac{f_M[(z_{M1} + z_{M2})(z_{M3} + z_{M4}) + z_{M3}z_{M4}]}{H_M} \qquad 4.71$$

$$\dot{\phi}_0 = \frac{f_R[(z_{R1} + z_{R2})(z_{R3} + z_{R4}) + z_{R3}z_{R4}]}{H_R} \qquad 4.72$$

$$\dot{X}_0 = \frac{p[(z_{A1} + z_{A2})(z_{A3} + z_{A4}) + z_{A3}z_{A4}]}{H_A} \qquad 4.73$$

where
$$z_{E1} = \frac{1}{j\omega C_{E1}} \qquad z_{R1} = \frac{1}{j\omega C_{R1}} \qquad 4.74 \quad 4.75$$

$$z_{E2} = j\omega L_1 \qquad z_{R2} = j\omega I_1 \qquad 4.76 \quad 4.77$$

$$z_{E3} = \frac{1}{j\omega C_{E2}} \qquad z_{R3} = \frac{1}{j\omega C_{R2}} \qquad 4.78 \quad 4.79$$

$$z_{E4} = r_E + j\omega L_2 \qquad z_{R2} = r_R + j\omega I_2 \qquad 4.80 \quad 4.81$$

$$z_{M1} = \frac{1}{j\omega C_{M1}} \qquad z_{A1} = \frac{1}{j\omega C_{A1}} \qquad 4.82 \quad 4.83$$

$$z_{M2} = j\omega m_1 \qquad z_{A2} = j\omega M_1 \qquad 4.84 \quad 4.85$$

$$z_{M3} = \frac{1}{j\omega C_{M2}} \qquad z_{A3} = \frac{1}{j\omega C_{A2}} \qquad 4.86 \quad 4.87$$

$$z_{M4} = r_M + j\omega m_2 \qquad z_{A4} = r_A + j\omega M_2 \qquad 4.88 \quad 4.89$$

$$H_E = z_{E1}z_{E3}z_{E4} + z_{E1}z_{E2}(z_{E3} + z_{E4}) \tag{4.90}$$

$$H_M = z_{M1}z_{M3}z_{M4} + z_{M1}z_{M2}(z_{M3} + z_{M4}) \tag{4.91}$$

$$H_R = z_{R1}z_{R3}z_{R4} + z_{R1}z_{R2}(z_{R3} + z_{R4}) \tag{4.92}$$

$$H_A = z_{A1}z_{A3}z_{A4} + z_{A1}z_{A2}(z_{A3} + z_{A4}) \tag{4.93}$$

The current $\dot{i}_1$, the linear velocity $\dot{x}_1$, the angular $\dot{\phi}_1$ and the volume current $\dot{x}_1$ are given by

$$\dot{i}_1 = \frac{e[z_{E2}(z_{E3} + z_{E4}) + z_{E3}z_{E4}]}{H_E} \tag{4.94}$$

$$\dot{x}_1 = \frac{f_M[z_{M2}(z_{M3} + z_{M4}) + z_{M3}z_{M4}]}{H_M} \tag{4.95}$$

$$\dot{\phi}_1 = \frac{f_R[z_{R2}(z_{R3} + z_{R4}) + z_{R3}z_{R4}]}{H_R} \tag{4.96}$$

$$\dot{X}_1 = \frac{p[z_{A2}(z_{A3} + z_{A4}) + z_{A3}z_{A4}]}{H_A} \tag{4.97}$$

The current $\dot{i}_2$, the linear velocity $\dot{x}_2$, the angular velocity $\dot{\phi}_2$, and the volume current $\dot{X}_2$ are given by

$$\dot{i}_2 = \frac{e z_{E1}(z_{E3} + z_{E4})}{H_E} \tag{4.98}$$

$$\dot{x}_2 = \frac{f_M z_{M1}(z_{M3} + z_{M4})}{H_M} \tag{4.99}$$

$$\dot{\phi}_2 = \frac{f_R z_{R1}(z_{R3} + z_{R4})}{H_R} \tag{4.100}$$

$$\dot{X}_2 = \frac{p z_{A1}(z_{A3} + z_{A4})}{H_A} \tag{4.101}$$

The current i_3, the linear velocity $\dot{x}_3$, the angular velocity $\dot{\phi}_3$ and the volume current $\dot{X}_3$ are given by

$$i_3 = \frac{e z_{E1} z_{E4}}{H_E} \qquad 4.102$$

$$\dot{x}_3 = \frac{f_M z_{M1} z_{M4}}{H_M} \qquad 4.103$$

$$\dot{\phi}_3 = \frac{f_R z_{R1} z_{R4}}{H_R} \qquad 4.104$$

$$\dot{X}_3 = \frac{p z_{A1} z_{A4}}{H_A} \qquad 4.105$$

The current i_4, the linear velocity $\dot{x}_4$, the angular velocity $\dot{\phi}_4$ and the volume current $\dot{X}_4$ are given by

$$i_4 = \frac{e z_{E1} z_{E3}}{H_E} \qquad 4.106$$

$$\dot{x}_4 = \frac{f_M z_{M1} z_{M3}}{H_M} \qquad 4.107$$

$$\dot{\phi}_4 = \frac{f_R z_{R1} z_{R3}}{H_R} \qquad 4.108$$

$$\dot{X}_4 = \frac{p z_{A1} z_{A3}}{H_A} \qquad 4.109$$

The equations in this section show that the equations for the electrical, mechanical rectilineal, mechanical rotational and acoustical systems are similar and analogous.

CHAPTER V

CORRECTIVE NETWORKS

5.1. Introduction

A corrective network is a structure which has a transmission characteristic that is more or less gradual in slope. Such a characteristic is obtained when an inductance, electrical capacitance or the combination of both is shunted across a line.[1] Another type of corrective network is an inductance, electrical capacitance or combination of both connected in series with a line. Resistance may be introduced as a second or third element in either shunt or series corrective networks. Various types of resistance networks may be used as attenuators or for matching dissimilar impedances. It is the purpose of this chapter to illustrate further analogies between electrical, mechanical rectilineal, mechanical rotational, and acoustical systems having similar transmission characteristics.

5.2. Two Electrical, Mechanical Rectilineal, Mechanical Rotational or Acoustical Impedances in Parallel

Two electrical impedances z_{E1} and z_{E2} are shown in parallel in Fig. 5.1. The electrical impedance z_{ET} of the two electrical impedances in parallel is given by

$$z_{ET} = \frac{z_{E1} z_{E2}}{z_{E1} + z_{E2}} \qquad 5.1$$

The electromotive force e across z_{ET} is also the electromotive force across z_{E1} and z_{E2}.

The total current i_T is the vector sum of the currents i_1 and i_2 as follows:

$$i_T = i_1 + i_2 \qquad 5.2$$

[1] The term "line" is used in this chapter to designate an electrical network which, prior to the introduction of the corrective network, consisted of a generator in series with two electrical impedances, termed the input and output electrical impedances.

If z_{E2} is made infinite the current i_2 in this branch is zero and the total current flows in z_{E1}, that is, $i_T = i_1$. In the same way if z_{E1} is made infinite the current i_1 in this branch is zero and the total current flows in z_{E2}, that is, $i_T = i_2$.

The mechanical rectilineal system, Fig. 5.1, consists of a system of rigid massless levers and links with frictionless bearings at the connecting

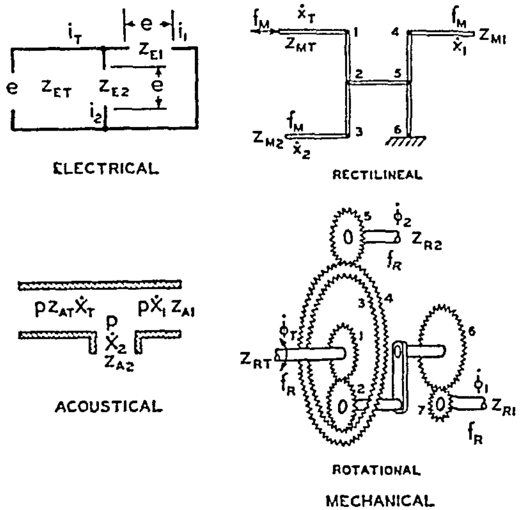

FIG. 5.1. Two electrical impedances connected in parallel and the mechanical rectilineal, mechanical rotational and acoustical analogies.

points. The member *1, 2, 3* is a lever or inflexible rod. The member *4, 5, 6* is also a lever or inflexible rod. The bearing *6* at the lower end of the lever *4, 5, 6* is connected to a rigid support. The driving force is connected by a link to point *1* on the lever *1, 2, 3*. The link *2, 5* connects the mid point of the lever *1, 2, 3* with the mid point of the lever *4, 5, 6*. A link connects the impedance z_{M2} to point *3* of the lever *1, 2, 3*. A link connects the impedance z_{M1} to *4* of the lever *4, 5, 6*.

Since the levers and links are massless and rigid the same force f_M exists at points *4* and *3* for driving the mechanical impedances z_{M1} and z_{M2} at these points. This is analogous to the same electromotive force across the impedances z_{E1} and z_{E2} in the electrical circuit.

The linear displacement x_T at *1* (for small amplitudes) is equal to the vector sum of the displacements x_1 and x_2 of points *3* and *4*, respectively.

$$x_T = x_1 + x_2 \tag{5.3}$$

Differentiating equation 5.3,

$$\dot{x}_T = \dot{x}_1 + \dot{x}_2 \tag{5.4}$$

That is, the linear velocity $\dot{x}_T$ at *1* is equal to the vector sum of the linear velocities $\dot{x}_1$ and $\dot{x}_2$ at points *4* and *3*, respectively. Equation 5.4 is analogous to equation 5.2 for the electrical system.

Since the same force f_M exists at points *3* and *4* as the driving point and further since the velocity at *1* is the vector sum of the velocities at points *3* and *4*, the mechanical impedances z_{M2} and z_{M1} at points *3* and *4*, respectively, must be in parallel. That is, the mechanical impedance z_{MT} at point *1* is

$$z_{MT} = \frac{z_{M1} z_{M2}}{z_{M1} + z_{M2}} \tag{5.5}$$

If z_{M2} is made infinite there can be no motion at point *3* and therefore the system behaves the same as if z_{M1} were connected to the point *1*. In the same way if z_{M1} is made infinite there is no motion at point *4* and the system behaves the same as if z_{M2} were connected directly to the point *1*.

The mechanical rotational system, Fig. 5.1, consists of a system of planetary [2] gears. The diameters of gears *1* and *2* are equal. The inside diameter of the large gear *3* is three times the gears *1* and *2*. The outside diameter of the gear *4* is three times that of gear *5*. The diameter of gear *6* is four times that of gear *7*. Gear *2* is free to rotate on its shaft.

[2] Practically all rotational systems which connect two mechanical rotational systems in parallel are of the epicyclic train type. In this book the planetary system is used to depict and illustrate a rotational system which connects mechanical rotational impedances in parallel. There are other examples of epicyclic trains which connect mechanical rotational impedances in parallel, as for example, the differential used in automobiles and tractors, shown schematically in Fig. 5.1*A*. The gear *2* is keyed to the shaft *1*. The gear *2* drives the ring gear *3*. The gears *4* and *9* rotate on bearings in the ring gear *3*. The ring gear *3* rotates freely on the shaft *8*. Gears *4* and *9* drive the gears *5* and *6*. Gears 5 and 6 are keyed to the shafts *7* and *8*, respectively. The diameter of gear *2* is one-half of the diameter of ring gear *3*. With these specifications the differential of Fig. 5.1*A* performs the same function as the planetary system of Fig. 5.1 with the same relations existing between f_R, $\dot{\phi}_T$, $\dot{\phi}_1$, $\dot{\phi}_2$, z_{RT}, z_{R1} and z_{R2} in both illustrations.

The large gear *3-4* is free to rotate with its axis coincident with gear *1*. The remainder of the gears are keyed to the respective shafts. Under these conditions if gear *7* is held stationary the angular displacement of gear *5* is the same as the driving gear *1*. Or if *5* is held stationary the angular displacement of gear *7* is the same as the driving gear *1*. In all the considerations which follow it is assumed that the ratios for the

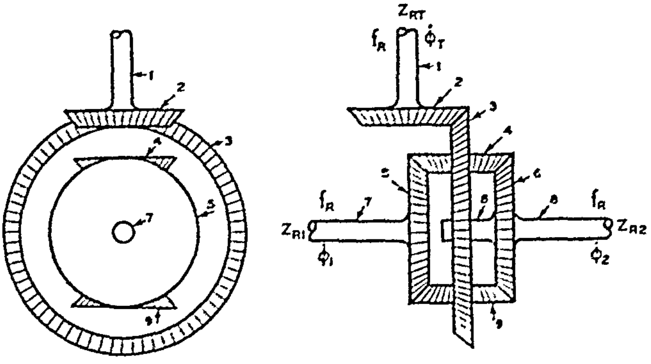

FIG. 5.1*A*. Differential gear train which connects two mechanical rotational impedances in parallel. This system accomplishes the same results as the planetary system of Fig. 5.1.

various gears are as outlined above. In addition it is assumed that all the gears are massless and that all the bearings are frictionless.

Since the gears are massless, the torque at gears *7* and *5* for driving the rotational impedances z_{R1} and z_{R2} is the same as the applied torque. This is analogous to the same electromotive force across the impedances z_{E1} and z_{E2} in the electrical circuit.

The angular displacement ϕ_T at gear *1* is equal to the vector sum of the angular displacement at ϕ_1 and ϕ_2 of the gears *7* and *5*, respectively.

$$\phi_T = \phi_1 + \phi_2 \tag{5.6}$$

Differentiating equation 5.6,

$$\dot{\phi}_T = \dot{\phi}_1 + \dot{\phi}_2 \tag{5.7}$$

That is, the angular velocity $\dot{\phi}_T$ at *1* is equal to the vector sum of the angular velocities $\dot{\phi}_2$ and $\dot{\phi}_1$ at the gears *5* and *7*, respectively. Equation 5.7 is analogous to equation 5.2 for the electrical system.

Since the same torque f_R exists at gears *5* and *7* as the driving point and, further, since the angular velocity at gear *1* is the vector sum of the angular velocities at gears *5* and *7*, the rotational impedances z_{R2} and z_{R1} at gears *7* and *5*, respectively, must be in parallel. That is, the mechanical rotational impedance z_{RT} at gear *1* is

$$z_{RT} = \frac{z_{R1}z_{R2}}{z_{R1} + z_{R2}} \qquad 5.8$$

If z_{R2} is made infinite there can be no motion at gear *5* and therefore the system behaves the same as if z_{R1} were connected to the shaft of gear *1*. In the same way, if z_{R1} is made infinite there is no motion at gear *7* and the system behaves the same as if z_{R2} were connected directly to the shaft of gear *1*.

The acoustical system of Fig. 5.1 consists of a three way connector. The dimensions are assumed to be small compared to the wavelength. Therefore, the pressure which actuates the two acoustical impedances is the same as the driving pressure. The total volume current $\dot{X}_T$ is the vector sum of the volume current $\dot{X}_1$ and $\dot{X}_2$, that is

$$\dot{X}_T = \dot{X}_1 + \dot{X}_2 \qquad 5.9$$

Equation 5.9 is analogous to equation 5.2 for the electrical system. The input acoustical impedance is

$$z_{AT} = \frac{z_{A1}z_{A2}}{z_{A1} + z_{A2}} \qquad 5.10$$

If z_{A2} is made infinite, there is no volume current in this branch and of course z_{AT} becomes z_{A1}. In the same way if z_{A1} is made infinite z_{AT} becomes z_{A2}. Thus it will be seen that the acoustical system of Fig. 5.1 is analogous to the electrical system of Fig. 5.1.

5.3. Shunt Corrective Networks

In a shunt corrective electrical network an electrical resistance, inductance, electrical capacitance or a combination of these elements is shunted across a line.[2]

[2] The term "line" is used to designate an electrical network which prior to the introduction of the corrective electrical network consisted of a generator in series with two electrical impedances z_{E1} and z_{E2}, termed the input and output electrical impedances. The shunt corrective electrical network z_{E3} is connected in parallel with the output electrical impedance z_{E2}.

The output current i_3 of a line shunted by an electrical network is given by

$$i_3 = \frac{e z_{E2}}{z_{E1}z_{E2} + z_{E1}z_{E3} + z_{E2}z_{E3}} \qquad 5.11$$

where z_{E1} = input electrical impedance,
z_{E2} = electrical impedance of the corrective electrical network,
z_{E3} = output electrical impedance, or the electrical impedance connected in shunt with the network, and
e = electromotive force in series with the input electrical impedance.

The output velocity $\dot{x}_3$ of a mechanical rectilineal network which is analogous to the shunt electrical network is given by

$$\dot{x}_3 = \frac{f_M z_{M2}}{z_{M1}z_{M2} + z_{M1}z_{M3} + z_{M2}z_{M3}} \qquad 5.12$$

where z_{M1} = input mechanical rectilineal impedance,
z_{M2} = mechanical rectilineal impedance of the corrective mechanical rectilineal network,
z_{M3} = output mechanical rectilineal impedance, and
f_M = mechanical driving force in series with the input mechanical rectilineal impedance.

The output angular velocity $\dot{\phi}_3$ of a mechanical rotational network which is analogous to the shunt electrical network is given by

$$\dot{\phi}_3 = \frac{f_R z_{R2}}{z_{R1}z_{R2} + z_{R1}z_{R3} + z_{R2}z_{R3}} \qquad 5.13$$

where z_{R1} = input mechanical rotational impedance,
z_{R2} = mechanical rotational impedance of the corrective mechanical rotational network,
z_{R3} = output mechanical rotational impedance, and
f_R = driving torque in series with the input rotational impedance.

The output volume current $\dot{X}_3$ of an acoustical network which is analogous to the shunt electrical network is given by

$$\dot{X}_3 = \frac{p z_{A2}}{z_{A1}z_{A2} + z_{A1}z_{A3} + z_{A2}z_{A3}} \qquad 5.14$$

where z_{A1} = input acoustical impedance,
z_{A2} = acoustical impedance of the corrective network,
z_{A3} = output acoustical impedance, and
p = driving pressure in series with the input acoustical impedance.

5.4. Inductance in Shunt with a Line and the Mechanical Rectilineal, Mechanical Rotational and Acoustical Analogies

In Fig. 5.2 an inductance is shunted across a line. The electrical impedance of an inductance is

$$z_{E2} = j\omega L \qquad 5.15$$

where $\omega = 2\pi f$,
f = frequency, in cycles per second, and
L = inducta ıce, in abhenries.

Equations 5.11 and 5.15 show that if the electrical impedance of the inductance is small compared to the input and output electrical impedances, the transmission will be small. If the electrical impedance of the inductance is large compared to the input and output electrical impedances, the attenuation introduced by the inductance will be negligible. Since the electrical impedance of an inductance is proportional to the frequency, the transmission will increase with frequency as shown by the characteristic [4] of Fig. 5.2.

The mechanical rectilineal impedance of the mass in Fig. 5.2 is

$$z_{M2} = j\omega m \qquad 5.16$$

where m = mass, in grams.

When the mass reactance in the mechanical rectilineal system of Fig. 5.2 is small compared to the load or driving mechanical rectilineal imped-

[4] The verbal description and the depicted transmission frequency characteristics in this chapter tacitly assume preponderately resistive input and output impedances. Of course the equations are valid for any kind of input and output impedances.

Since the mechanical rotational impedance of a moment of inertia is proportional to the frequency, the transmission will increase with frequency as shown by the characteristic of Fig. 5.2.

The acoustical system of Fig. 5.2 consists of a pipe with a side branch forming an inertance. The acoustical impedance of the inertance in Fig. 5.2 is

$$z_{A2} = j\omega M \tag{5.18}$$

where M = inertance, in grams per (centimeter)^4.

Equations 5.14 and 5.18 show that at low frequencies the acoustical reactance of the inertance is small compared to the acoustical impedance of the pipe and the sound is shunted out through the hole. At high frequencies the acoustical reactance of the inertance is large compared to the acoustical impedance of the pipe and the sound wave flows down the pipe the same as it would in the absence of a branch. Since the acoustical impedance of an inertance is proportional to the frequency, the transmission will increase with frequency as shown by the characteristic of Fig. 5.2.

5.5. Electrical Capacitance in Shunt with a Line and the Mechanical Rectilineal, Mechanical Rotational and Acoustical Analogies

In Fig. 5.3 an electrical capacitance is shunted across a line. The electrical impedance of an electrical capacitance is

$$z_{E2} = \frac{1}{j\omega C_E} \tag{5.19}$$

where C_E = electrical capacitance, in abfarads.

The electrical reactance of an electrical capacitance is inversely proportional to the frequency. Therefore, from equations 5.11 and 5.19 the transmission will decrease with increase in frequency as shown by the characteristic of Fig. 5.3.

The mechanical rectilineal impedance of the compliance in Fig. 5.3 is

$$z_{M2} = \frac{1}{j\omega C_M} \tag{5.20}$$

where C_M = compliance, in centimeters per dyne.

The mechanical rectilineal reactance of the compliance of the rectilineal system, Fig. 5.3, is inversely proportional to the frequency. Equations 5.12 and 5.20 show that at low frequencies the velocity at the input to the compliance will be small and the behavior will be practically the same as that of a directly coupled system. At high frequencies the velocity of the compliance will be practically the same as the input

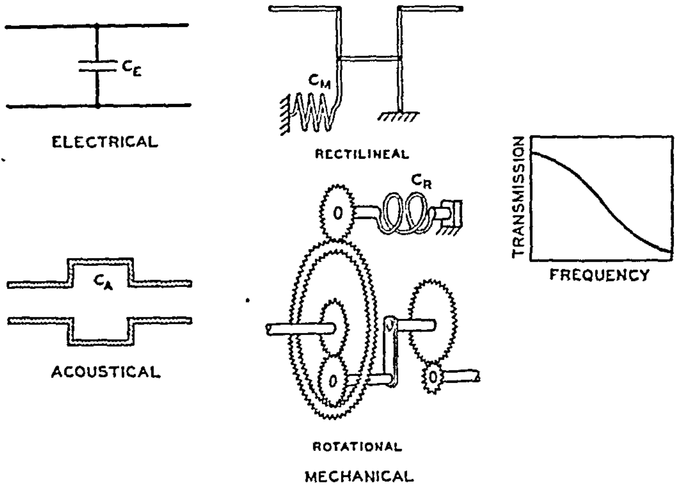

FIG. 5.3. Electrical capacitance in shunt with a line and the mechanical rectilineal, mechanical rotational and acoustical analogies. The graph depicts the transmission frequency characteristic.

velocity and there will be very little velocity transmitted to the load. The transmission characteristic of this system obtained from equations 5.12 and 5.20 is shown in Fig. 5.3.

The mechanical rotational impedance of the rotational compliance of Fig. 5.3 is

$$z_{R2} = \frac{1}{j\omega C_R} \qquad 5.21$$

where C_R = rotational compliance, in radians per dyne per centimeter.

The mechanical rotational impedance of the rotational compliance of the mechanical rotational system of Fig. 5.3 is inversely proportional to the frequency. Equations 5.13 and 5.21 show that at low frequencies

the angular velocity at the input to the rotational compliance will be small and the behavior will be practically the same as that of a directly coupled system. At high frequencies the angular velocity of the rotational compliance will be the same as the input angular velocity and there will be very little angular velocity transmitted to the load. The transmission characteristic of this system obtained from equations 5.13 and 5.21 is shown in Fig. 5.3.

The acoustical system of Fig. 5.3 consists of a pipe with an enlarged portion forming an acoustical capacitance. The acoustical impedance of an acoustical capacitance is

$$z_{A2} = \frac{1}{j\omega C_A} \qquad 5.22$$

where C_A = acoustical capacitance, in (centimeter)^5 per dyne.

At low frequencies the acoustical reactance of the acoustical capacitance is large compared to the impedance of the pipe and the sound flows down the pipe the same as it would in the absence of the enlargement. At high frequencies the acoustical reactance of the acoustical capacitance is small compared to the acoustical impedance of the pipe and the sound is shunted out by the enlargement. Since the acoustical reactance is inversely proportional to the frequency, equations 5.14 and 5.22 show that the transmission will decrease with frequency as shown by the characteristic of Fig. 5.3.

5.6. Inductance and Electrical Capacitance in Series, in Shunt with a Line and the Mechanical Rectilineal, Mechanical Rotational and Acoustical Analogies

Fig. 5.4 shows an inductance and electrical capacitance connected in series across a line. The mechanical rectilineal, mechanical rotational, and acoustical analogies are also shown in Fig. 5.4.

The electrical impedance of the electrical network is

$$z_{E2} = j\omega L + \frac{1}{j\omega C_E} \qquad 5.23$$

where L = inductance, in abhenries, and
C_E = electrical capacitance, in abfarads.

The output current can be obtained from equations 5.11 and 5.23.

The mechanical rectilineal impedance of the mechanical rectilineal system is

$$z_{M2} = j\omega m + \frac{1}{j\omega C_M} \tag{5.24}$$

where m = mass, in grams, and
C_M = compliance, in centimeters per dyne.

The output velocity can be obtained from equations 5.12 and 5.24.

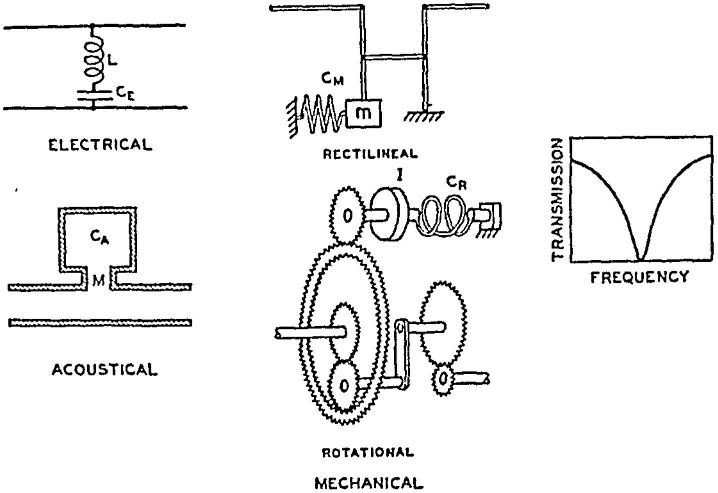

FIG. 5.4. Inductance and electrical capacitance in series, in shunt with a line and the mechanical rectilineal, mechanical rotational and acoustical analogies. The graph depicts the transmission frequency characteristic.

The mechanical rotational impedance of the mechanical rotational system is

$$z_{R2} = j\omega I + \frac{1}{j\omega C_R} \tag{5.25}$$

where I = moment of inertia, in gram (centimeter)^2, and
C_R = rotational compliance, in radians per dyne per centimeter.

The output angular velocity can be obtained from equations 5.13 and 5.25.

The acoustical impedance of the acoustical system is

$$z_{A2} = j\omega M + \frac{1}{j\omega C_A} \tag{5.26}$$

where M = inertance, in grams per (centimeter)^4, and
C_A = acoustical capacitance, in (centimeter)^5 per dyne.

The output volume current can be obtained from equations 5.14 and 5.26.

At low frequencies the four systems behave the same as those of Fig. 5.3 and there is very little attenuation. At high frequencies the systems behave the same as those of Fig. 5.2 and there is very little attenuation. At the resonant frequency of the inductance and the electrical capacitance the electrical impedance is zero and equations 5.11 and 5.23 show that there is no transmission at this frequency. At the resonant frequency of the mass and compliance no motion is transmitted because the force required to drive the resonant system is zero. Equations 5.12 and 5.24 also show that there is no transmission at the resonant frequency of the mass and compliance. At the resonant frequency of the moment of inertia and rotational compliance no angular motion is transmitted because the torque required to actuate the resonant system is zero. Equations 5.13 and 5.25 also show that there is no transmission at the resonant frequency of the moment of inertia and rotational compliance. At the resonant frequency of the inertance and acoustical capacitance there will be no transmission because the pressure at the input to the resonator is zero. Equations 5.14 and 5.26 also show that there is no transmission at the resonant frequency of the inertance and acoustical capacitance. The transmission characteristics of the four systems are shown in Fig. 5.4.

5.7. Inductance and Electrical Capacitance in Parallel, in Shunt with a Line and the Mechanical Rectilineal, Mechanical Rotational and Acoustical Analogies

Fig. 5.5 shows an inductance and electrical capacitance connected in parallel across a line. The mechanical rectilineal, mechanical rotational and acoustical equivalents are also shown in Fig. 5.5.

The electrical impedance of the electrical network of Fig. 5.5 is

$$z_{E2} = \frac{j\omega L}{1 - \omega^2 L C_E} \tag{5.27}$$

where L = inductance, in abhenries, and
C_E = electrical capacitance, in abfarads.

The output current can be obtained from equations 5.11 and 5.27.

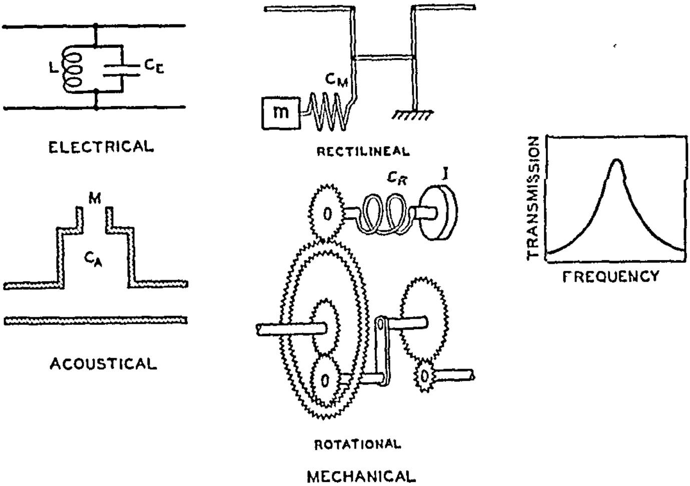

Fig. 5.5. Inductance and electrical capacitance in parallel, in shunt with a line and the mechanical rectilineal, mechanical rotational and acoustical analogies. The graph depicts the transmission frequency characteristic.

The mechanical rectilineal impedance of the mechanical rectilineal system of Fig. 5.5 is

$$z_{M2} = \frac{j\omega m}{1 - \omega^2 m C_M} \tag{5.28}$$

where m = mass, in grams, and
C_M = compliance, in centimeters per dyne.

The output velocity can be obtained from equations 5.12 and 5.28.

The mechanical rotational impedance of the mechanical rotational system of Fig. 5.5 is

$$z_{R2} = \frac{j\omega I}{1 - \omega^2 I C_R} \qquad 5.29$$

where I = moment of inertia, in gram (centimeter)^2, and
C_R = rotational compliance, in radians per dyne per centimeter.

The output angular velocity can be obtained from equations 5.13 and 5.29.

The acoustical impedance of the acoustical system of Fig. 5.5 is

$$z_{A2} = \frac{j\omega M}{1 - \omega^2 M C_A} \qquad 5.30$$

where M = inertance, in grams per (centimeter)^4, and
C_A = acoustical capacitance, in (centimeter)^5 per dyne.

The output volume current can be obtained from equations 5.14 and 5.30.

At low frequencies the systems behave the same as those of Fig. 5.2 and the transmission is small. At high frequencies the systems behave the same as those of Fig. 5.3 and the transmission is again small. At the resonant frequency of the inductance and electrical capacitance the electrical impedance is infinite and equations 5.11 and 5.27 show that the shunt circuit introduces no attenuation at the resonant frequency. At the resonant frequency of the mass and compliance the input to the spring does not move because the mechanical rectilineal impedance is infinite and the behavior is the same as a directly coupled system. Equations 5.12 and 5.28 also show that there is no attenuation due to the shunt system at the resonant frequency of the mass and compliance. At the resonant frequency of the moment of inertia with the rotational compliance the input to the spring will not turn because the mechanical rotational impedance is infinite and the behavior is the same as a directly coupled system. Equations 5.13 and 5.29 also show that there is no attenuation due to the shunt system at the resonant frequency. At the resonant frequency of the inertance and acoustical capacitance the input volume current is zero because the acoustical impedance is infinite and the behavior is the same as a plain pipe. Equations 5.14 and 5.30 also show that there is no attenuation due to the shunt system at the reso-

nant frequency. The transmission characteristics of the four systems are shown in Fig. 5.5.

5.8. Electrical Resistance, Inductance and Electrical Capacitance in Series, in Shunt with a Line and the Mechanical Rectilineal, Mechanical Rotational and Acoustical Analogies

Fig. 5.6 shows an electrical resistance, inductance and electrical capacitance in series, in shunt with a line. The mechanical rectilineal, mechan-

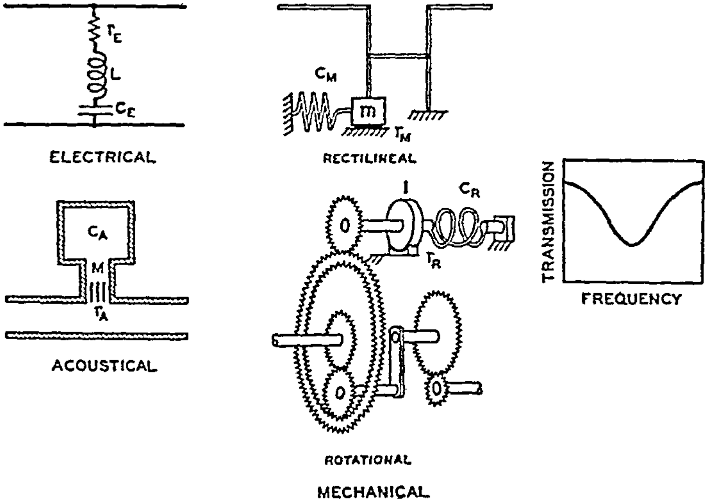

Fig. 5.6. Electrical resistance, inductance and electrical capacitance in series, in shunt with a line and the mechanical rectilineal, mechanical rotational and acoustical analogies. The graph depicts the transmission frequency characteristic.

ical rotational and acoustical analogies are also shown in Fig. 5.6. The electrical impedance of the electrical network is

$$z_{E2} = r_E + j\omega L + \frac{1}{j\omega C_E} \qquad 5.31$$

where r_E = electrical resistance, in abohms,
L = inductance, in abhenries, and
C_E = electrical capacitance, in abfarads.

The output current can be obtained from equations 5.11 and 5.31.

The mechanical rectilineal impedance of the mechanical rectilineal system is

$$z_{M2} = r_M + j\omega m + \frac{1}{j\omega C_M} \qquad 5.32$$

where r_M = mechanical rectilineal resistance, in mechanical ohms,
m = mass, in grams,
C_M = compliance, in centimeters per dyne.

The output velocity can be obtained from equations 5.12 and 5.32.

The mechanical rotational impedance of the mechanical rotational system is

$$z_{R2} = r_R + j\omega I + \frac{1}{j\omega C_R} \qquad 5.33$$

where r_R = mechanical rotational resistance, in rotational ohms,
I = moment of inertia, in gram (centimeter)^2, and
C_R = rotational compliance, in radians per dyne per centimeter.

The output rotational velocity can be obtained from equations 5.13 and 5.33.

The acoustical impedance of the acoustical system is

$$z_{A2} = r_A + j\omega M + \frac{1}{j\omega C_A} \qquad 5.34$$

where r_A = acoustical resistance, in acoustical ohms,
M = inertance, in grams per (centimeter)^4, and
C_A = acoustical capacitance, in (centimeter)^5 per dyne.

The output volume current can be obtained from equations 5.14 and 5.34.

At low frequencies the systems behave the same as those of Fig. 5.3 and there is very little attenuation. At high frequencies the systems behave the same as those of Fig. 5.2 and there is very little attenuation. At the resonant frequency of the inductance and electrical capacitance the electrical reactance is zero. Therefore, from equations 5.11 and 5.31 the transmission as influenced by the network is governed by the electrical resistance. At the resonant frequency of the mass and compliance the mechanical rectilineal reactance is zero. Therefore, from equations 5.12 and 5.32 the transmission as influenced by the mechanical rectilineal

system is governed by the mechanical rectilineal resistance. At the resonant frequency of the moment of inertia and rotational compliance the mechanical rotational reactance is zero. Therefore, from equations 5.13 and 5.33 the transmission as influenced by the mechanical rotational system is governed by the mechanical rotational resistance. At the resonant frequency of the inertance and acoustical capacitance the acoustical reactance is zero. Therefore, from equations 5.14 and 5.34 the transmission as influenced by the acoustical system is governed by the acoustical resistance. The transmission characteristic of these systems is shown in Fig. 5.6. This characteristic at the high and low frequencies is the same as that of Fig. 5.4. However, in the region of resonance the resistance in each of the four systems decreases the attenuation as depicted by the transmission characteristic of Fig. 5.6.

5.9. Electrical Resistance, Inductance and Electrical Capacitance in Parallel, in Shunt with a Line and the Mechanical Rectilineal, Mechanical Rotational and Acoustical Analogies

Fig. 5.7 shows an electrical resistance, inductance and electrical capacitance connected in parallel across a line. The mechanical rectilineal, mechanical rotational and acoustical analogies are also shown in Fig. 5.7. The electrical impedance of the electrical network is

$$z_{E2} = \frac{j\omega r_E L}{r_E - \omega^2 r_E L C_E + j\omega L} \qquad 5.35$$

where r_E = electrical resistance, in abohms,
L = inductance, in abhenries, and
C_E = electrical capacitance, in abfarads.

The output current can be obtained from equations 5.11 and 5.35.
The mechanical rectilineal impedance of the mechanical system is

$$z_{M2} = \frac{j\omega r_M m}{r_M - \omega^2 r_M m C_M + j\omega m} \qquad 5.36$$

where r_M = mechanical rectilineal resistance, in mechanical ohms,
m = mass, in grams, and
C_M = compliance, in centimeters per dyne.

The output velocity can be obtained from equations 5.12 and 5.36.

The mechanical rotational impedance of the rotational system is

$$z_{R2} = \frac{j\omega r_R I}{r_R - \omega^2 r_R I C_R + j\omega I} \qquad 5.37$$

where r_R = mechanical rotational resistance, in rotational ohms,
I = moment of inertia, in gram (centimeter)^2, and
C_R = rotational compliance, in radians per dyne per centimeter.

The output rotational velocity can be obtained from equations 5.13 and 5.37.

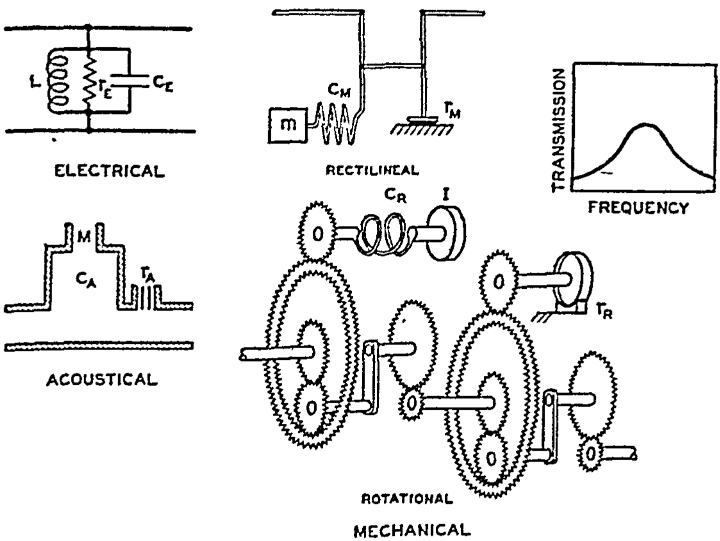

Fig. 5.7. Electrical resistance, inductance and electrical capacitance in parallel, in shunt with a line and the mechanical rectilineal, mechanical rotational and acoustical analogies. The graph depicts the transmission frequency characteristic.

The acoustical impedance of the acoustical system is

$$z_{A2} = \frac{j\omega r_A M}{r_A - \omega^2 r_A M C_A + j\omega M} \qquad 5.38$$

where r_A = acoustical resistance, in acoustical ohms,
M = inertance, in grams per (centimeter)^4, and
C_A = acoustical capacitance, in (centimeter)^5 per dyne.

The output volume current can be obtained from equations 5.14 and 5.38.

At low frequencies the systems behave the same as those of Fig. 5.2 and the transmission is small. At high frequencies the systems behave the same as those of Fig. 5.3 and the transmission is again small. At the resonant frequency the electrical reactance is infinite and therefore from equations 5.11 and 5.35 the attenuation is due to the electrical resistance. At the resonant frequency of the mass and compliance, Fig. 5.7, the mechanical rectilineal reactance is infinite and therefore from equations 5.12 and 5.36 the attenuation is due to the mechanical rectilineal resistance. At the resonant frequency of the moment of inertia and rotational compliance, Fig. 5.7, the mechanical rotational reactance is infinite and from equations 5.13 and 5.37 the attenuation is due to the mechanical rotational resistance. At the resonant frequency of the inertance and acoustical capacitance, Fig. 5.7, the acoustical reactance is infinite and from equations 5.14 and 5.38 the attenuation is due to the acoustical resistance. The transmission characteristic of these systems is shown in Fig. 5.7. This characteristic at the low and high frequencies is the same as that of Fig. 5.5. However, in the region of resonance the resistance in each of the four systems introduces attenuation as depicted by the transmission characteristic of Fig. 5.7.

5.10. Series Corrective Networks

In a series electrical network an electrical resistance, inductance, electrical capacitance or a combination of these elements is connected in series with a line.[5]

The output current i_3 of a line containing a series electrical network is given by

$$i_3 = \frac{e}{z_{E1} + z_{E2} + z_{E3}} \qquad 5.39$$

where z_{E1} = input electrical impedance,
z_{E2} = electrical impedance of the corrective electrical network,
z_{E3} = output electrical impedance, and
e = electromotive force in series with the three electrical impedances.

[5] The term "line" is used to designate an electrical network which prior to the introduction of the corrective electrical network consisted of a generator in series with two electrical impedances z_{E1} and z_{E3}, termed the input and output electrical impedances. The series corrective electrical network z_{E2} is connected in series with the input and output electrical impedances z_{E1} and z_{E3}.

The output velocity $\dot{x}_3$ of a mechanical rectilineal network which is analogous to the series electrical network is given by

$$\dot{x}_3 = \frac{f_M}{z_{M1} + z_{M2} + z_{M3}} \qquad 5.40$$

where z_{M1} = input mechanical rectilineal impedance,
z_{M2} = mechanical rectilineal impedance of the corrective mechanical rectilineal network,
z_{M3} = output mechanical rectilineal impedance, and
f_M = mechanical driving force in series with mechanical impedances.

The output angular velocity $\dot{\phi}_3$ of a mechanical rotational network which is analogous to the series electrical network is given by

$$\dot{\phi}_3 = \frac{f_R}{z_{R1} + z_{R2} + z_{R3}} \qquad 5.41$$

where z_{R1} = input mechanical rotational impedance,
z_{R2} = mechanical rotational impedance of the corrective mechanical rotational network,
z_{R3} = output mechanical rotational impedance, and
f_R = rotational driving torque in series with the mechanical impedances.

The output volume current $\dot{X}_3$ of an acoustical network which is analogous to the series electrical network is given by

$$\dot{X}_3 = \frac{p}{z_{A1} + z_{A2} + z_{A3}} \qquad 5.42$$

where z_{A1} = input acoustical impedance,
z_{A2} = acoustical impedance of the corrective acoustical network,
z_{A3} = output acoustical impedance, and
p = driving pressure in series with the acoustical impedances.

5.11. Inductance in Series with a Line and the Mechanical Rectilineal, Mechanical Rotational and Acoustical Analogies

In Fig. 5.8 an inductance is connected in series with a line. The electrical impedance of the inductance is

$$z_{E2} = j\omega L \qquad 5.43$$

where L = inductance, in abhenries.

Equations 5.39 and 5.43 show that if the electrical impedance of the inductance is small compared to the input and output electrical impedances the attenuation introduced by the inductance will be small. If the electrical impedance of the inductance is large compared to the input and output electrical impedances the current transmission will be small. Since the electrical impedance of an inductance is proportional to the frequency, the transmission will decrease with frequency as shown in Fig. 5.8.

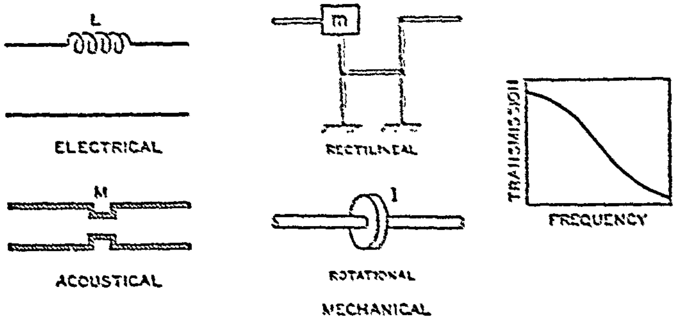

Fig. 5.8. Inductance in series with a line and the mechanical rectilineal, mechanical rotational and acoustical analogies. The graph depicts the transmission frequency characteristic.

The mechanical rectilineal impedance of the mass in Fig. 5.8 is

$$z_{M2} = j\omega m \qquad 5.44$$

where m = mass, in grams.

In the mechanical rectilineal system of Fig. 5.8 equations 5.40 and 5.44 show that if the mass reactance is small compared to the load or driving mechanical rectilineal impedance the addition of the mass will cause very little reduction in the velocity transmitted to the load. If the mass reactance is comparatively large the mass will remain practically stationary and the velocity transmitted to the load will be small. Since the mechanical rectilineal impedance of a mass is proportional to the frequency, the transmission will decrease with frequency as shown by the characteristic of Fig. 5.8.

The mechanical rotational impedance of the flywheel in Fig. 5.8 is

$$z_{R2} = j\omega I \qquad 5.45$$

where I = moment of inertia, in gram (centimeters)^2.

In the mechanical rotational system of Fig. 5.8 equations 5.41 and 5.45 show that if the moment of inertia reactance is small compared to the load or driving mechanical rectilineal impedance the addition of the moment of inertia will cause very little reduction in the angular velocity transmitted to the load. If the moment of inertia reactance is comparatively large the flywheel will remain practically stationary and the velocity transmitted to the load will be small. Since the mechanical rotational impedance of a moment of inertia is proportional to the frequency, the transmission will decrease with frequency as shown by the characteristic of Fig. 5.48.

The acoustical system of Fig. 5.8 consists of a pipe with a constriction which forms an inertance. The acoustical impedance of the inertance in Fig. 5.8 is

$$z_{A2} = j\omega M \tag{5.46}$$

where M = *inertance, in grams per (centimeter)*^{4}.

Equations 5.42 and 5.46 show that at the low frequencies where the acoustical impedance of an inertance is small compared to input and output acoustical impedances the attenuation introduced by the inertance is small. At the high frequencies the acoustical impedance of the inertance is large and the transmission is small. Since the acoustical impedance of an inertance is proportional to the frequency the transmission will decrease with frequency as shown by the characteristic of Fig. 5.8.

5.12. Electrical Capacitance in Series with a Line and the Mechanical Rectilineal, Mechanical Rotational and Acoustical Analogies

In Fig. 5.9 an electrical capacitance is connected in series with a line. The electrical impedance of the electrical capacitance of Fig. 5.9 is

$$z_{E2} = \frac{1}{j\omega C_E} \tag{5.47}$$

where C_E = electrical capacitance, in abfarads.

Equation 5.39 shows that if the electrical impedance of the electrical capacitance is large compared to the input or output electrical impedances, the attenuation introduced by the electrical capacitance will be large. If the electrical impedance of the electrical capacitance is small compared to the input and output electrical impedances the attenuation

will be small. Since the electrical impedance of an electrical capacitance is inversely proportional to the frequency, as shown by equation 5.47, the transmission will increase with frequency as shown in Fig. 5.9.

The mechanical rectilineal impedance of Fig. 5.9 is

$$z_{M2} = \frac{1}{j\omega C_M} \tag{5.48}$$

where C_M = compliance, in centimeters per dyne.

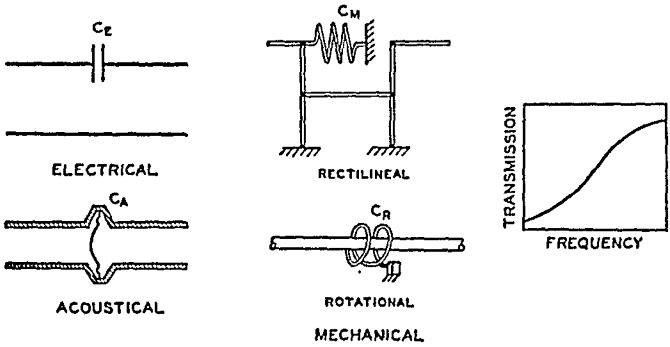

Fig. 5.9. Electrical capacitance in series with a line and the mechanical rectilineal, mechanical rotational and acoustical analogies. The graph depicts the transmission frequency characteristic.

Equation 5.48 shows that the mechanical impedance of the compliance of the mechanical rectilineal system of Fig. 5.9 is inversely proportional to the frequency. Equations 5.40 and 5.48 show that at low frequencies the input velocity to the compliance is relatively small and there will be little transmission. At high frequencies the input velocity to the compliance is relatively large and therefore it introduces very little impedance to motion. Therefore the transmission characteristic will be shown in Fig. 5.9.

The mechanical rotational impedance of Fig. 5.9 is

$$z_{R2} = \frac{1}{j\omega C_R} \tag{5.49}$$

where C_R = rotational compliance, in radians per dyne per centimeter.

Equation 5.49 shows that the mechanical rotational impedance of the rotational compliance of the mechanical rotational system of Fig. 5.9 is inversely proportional to the frequency. Equations 5.41 and 5.48 show that at the low frequencies the input angular velocity to the rotational compliance is relatively small and the transmission is low. At the high frequencies the angular velocity of the rotational compliance is relatively large and it introduces very little impedance to motion. Therefore the transmission characteristic will be as shown in Fig. 5.9.

There is no simple purely acoustical system which is analogous to an electrical capacitance in series with a line. The analogy shown in Fig. 5.9 consists of a stiffness controlled diaphragm, that is, the mass of the diaphragm is small and the stiffness high so that the frequency range under consideration is well below the natural resonant frequency of the diaphragm and suspension. The acoustical capacitance of this system is

$$C_A = C_M S^2 \tag{5.50}$$

where C_A = acoustical capacitance, in (centimeter)^5 per dyne,
C_M = compliance of the suspension system, in centimeters per dyne, and
S = area of the diaphragm, in square centimeters.

The acoustical impedance of Fig. 5.9 is

$$z_{A2} = \frac{1}{j\omega C_A} \tag{5.51}$$

where C_A = acoustical capacitance of equation 5.50.

It will be seen that this system will not transmit a steady flow of a gas in the same way that the electrical circuit of Fig. 5.9 will not transmit direct current. Equation 5.51 shows that the acoustical impedance of an acoustical capacitance is inversely proportional to the frequency. Equations 5.42 and 5.51 show that the transmission will increase with increase of the frequency as shown by the characteristic of Fig. 5.9.

5.13. Inductance and Electrical Capacitance in Series with a Line and the Mechanical Rectilineal, Mechanical Rotational, and Acoustical Analogies

Fig. 5.10 shows an inductance and acoustical capacitance connected in series with a line. The mechanical rectilineal, mechanical rotational and acoustical analogies are also shown in Fig. 5.10.

The electrical impedance of the electrical network of Fig. 5.10 is

$$z_{E2} = j\omega L + \frac{1}{j\omega C_E} \qquad 5.52$$

where L = inductance, in abhenries, and
C_E = electrical capacitance, in abfarads.

The output current can be obtained from equations 5.39 and 5.52.

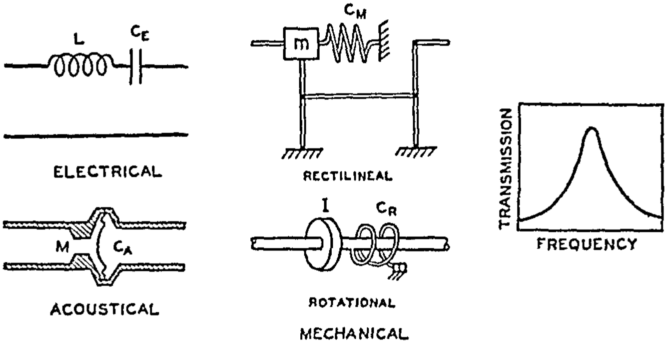

FIG. 5.10. Inductance and electrical capacitance in series with a line and the mechanical rectilineal, mechanical rotational and acoustical analogies. The graph depicts the transmission frequency characteristic.

The mechanical rectilineal impedance of the mechanical rectilineal system of Fig. 5.10 is

$$z_{M2} = j\omega m + \frac{1}{j\omega C_M} \qquad 5.53$$

where m = mass, in grams, and
C_M = compliance, in centimeters per dyne.

The output velocity can be obtained from equations 5.40 and 5.53.

The mechanical rotational impedance of the mechanical rotational system of Fig. 5.10 is

$$z_{R2} = j\omega I + \frac{1}{j\omega C_R} \qquad 5.54$$

where I = moment of inertia, in gram (centimeter)^2, and
C_R = rotational compliance, in radians per dyne per centimeter.

The output angular velocity can be obtained from equations 5.41 and 5.54.

The acoustical impedance of the acoustical system of Fig. 5.10 is

$$z_{A2} = j\omega M + \frac{1}{j\omega C_A} \tag{5.55}$$

where M = inertance, in grams per (centimeter)^4, and
C_A = acoustical capacitance, in (centimeter)^5 per dyne.

The expression for the acoustical capacitance is given by equation 5.50.

The output volume current can be obtained from equations 5.42 and 5.55.

At low frequencies the four systems behave the same as those of Fig. 5.9 and the transmission is small. At high frequencies the systems behave the same as those of Fig. 5.8, and again the transmission is small. At the resonant frequency of the inductance and electrical capacitance the electrical impedance of these two elements in series is zero and equations 5.39 and 5.52 show that the series circuit introduces no attenuation. At the resonant frequency of the mass and compliance the force required to drive the resonant system is zero and therefore the attenuation introduced by the system is zero. Equations 5.40 and 5.53 also show that the attenuation is zero at the resonant frequency of the mass and compliance. At the resonant frequency of the moment of inertia and rotational compliance the torque required to drive the resonant system is zero and therefore the attenuation introduced by the system is zero. Equations 5.41 and 5.54 also show that the attenuation is zero at the resonant frequency of the moment of inertia and rotational compliance. At the resonant frequency of the inertance and acoustical capacitance the pressure required to actuate the resonant system is zero. Equations 5.42 and 5.55 also show that the attenuation introduced by the system is zero. The transmission characteristics of the four systems are shown in Fig. 5.10.

5.14. Inductance and Electrical Capacitance in Parallel, in Series with a Line and the Mechanical Rectilineal, Mechanical Rotational and Acoustical Analogies

Fig. 5.11 shows an inductance and electrical capacitance in parallel connected in series with a line. The mechanical rectilineal, mechanical rotational and acoustical analogies are also shown in Fig. 5.11.

The electrical impedance of the electrical network of Fig. 5.11 is

$$z_{E2} = \frac{j\omega L}{1 - \omega^2 LC_E} \tag{5.56}$$

where L = inductance, in abhenries, and
C_E = electrical capacitance, in abfarads.

The output current can be obtained from equations 5.39 and 5.56.

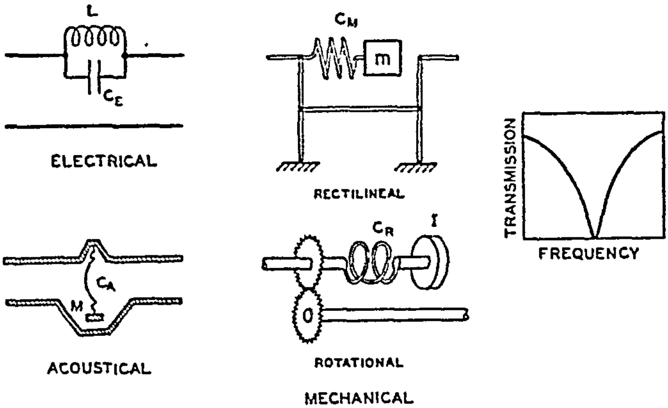

FIG. 5.11. Inductance and electrical capacitance in parallel, in series with a line and the mechanical rectilineal, mechanical rotational and acoustical analogies. The graph depicts the transmission frequency characteristic.

The mechanical rectilineal impedance of the mechanical rectilineal system of Fig. 5.11 is

$$z_{M2} = \frac{j\omega m}{1 - \omega^2 mC_M} \tag{5.57}$$

where m = mass, in grams, and
C_M = compliance, in centimeters per dyne.

The output velocity can be obtained from equations 5.40 and 5.57.

The mechanical rotational impedance of the mechanical rotational system of Fig. 5.11 is

$$z_{R2} = \frac{j\omega I}{1 - \omega^2 IC_R} \tag{5.58}$$

The electrical impedance of the electrical network of Fig. 5.12 is

$$z_{E2} = r_E + j\omega L + \frac{1}{j\omega C_E} \tag{5.60}$$

where r_E = electrical resistance, in abohms,
L = inductance, in abhenries, and
C_E = capacitance, in abfarads.

The output current can be obtained from equations 5.39 and 5.60.

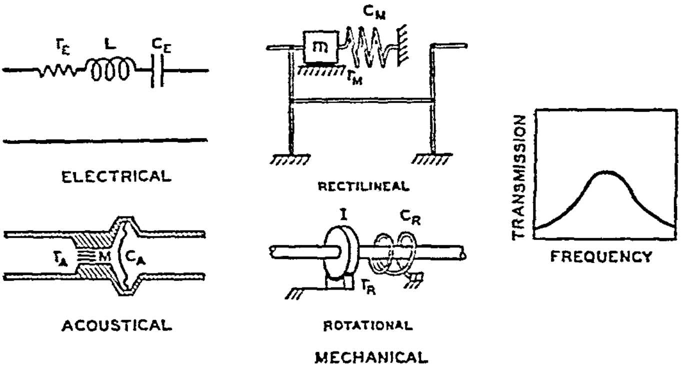

FIG. 5.12. Electrical resistance, inductance and electrical capacitance in series with a line and the mechanical rectilineal, mechanical rotational and acoustical analogies. The graph depicts the transmission frequency characteristic.

The mechanical rectilineal impedance of the mechanical system of Fig. 5.12 is

$$z_{M2} = r_M + j\omega m + \frac{1}{j\omega C_M} \tag{5.61}$$

where r_M = mechanical rectilineal resistance, in mechanical ohms,
m = mass, in grams,
C_M = compliance, in centimeters per dyne.

The output velocity can be obtained from equations 5.40 and 5.61.

The mechanical rotational impedance of the mechanical rotational system is

$$z_{R2} = r_R + j\omega I + \frac{1}{j\omega C_R} \tag{5.62}$$

where r_R = mechanical rotational resistance, in rotational ohms,
I = moment of inertia, in gram (centimeter)^2, and
C_R = rotational compliance, in radians per dyne per centimeter.

The output rotational velocity can be obtained from equations 5.41 and 5.62.

The acoustical impedance of the acoustical system of Fig. 5.12 is

$$z_{A2} = r_A + j\omega M + \frac{1}{j\omega C_A} \qquad 5.63$$

where r_A = acoustical resistance, in acoustical ohms,
M = inertance, in grams per (centimeter)^4, and
C_A = acoustical capacitance, in (centimeter)^5 per dyne.

The output volume current can be obtained from equations 5.42 and 5.63.

At low frequencies the four systems behave the same as those of Fig. 5.9 and the transmission is small. At high frequencies the systems behave the same as those of Fig. 5.8, and again the transmission is small. At the resonant frequency of the inductance and electrical capacitance the series electrical reactance is zero. Therefore, from equations 5.39 and 5.60 the attenuation is due to the electrical resistance. At the resonant frequency of the mass and compliance, Fig. 5.12, the mechanical rectilineal reactance is zero. Therefore, from equations 5.40 and 5.61 the attenuation is due to the mechanical resistance. At the resonant frequency of the moment of inertia and the rotational compliance, Fig. 5.12, the mechanical rotational reactance is zero. Therefore, from equations 5.41 and 5.62 the attenuation is due to the mechanical rotational resistance. At the resonant frequency of the inertance and acoustical capacitance, Fig. 5.12, the acoustical reactance is zero. Therefore, from equations 5.42 and 5.63 the attenuation is due to the acoustical resistance. The transmission characteristic of these systems is shown in Fig. 5.12. This characteristic at the low and high frequencies is the same as that of Fig. 5.10. However, in the region of resonance the resistance in each of the four systems introduces attenuation as depicted by the transmission characteristic of Fig. 5.12.

5.16. Electrical Resistance, Inductance and Electrical Capacitance in Parallel, in Series with a Line and the Mechanical Rectilineal, Mechanical Rotational and Acoustical Analogies

Fig. 5.13 shows an electrical resistance, inductance and electrical capacitance in parallel in series with a line. The mechanical rectilineal,

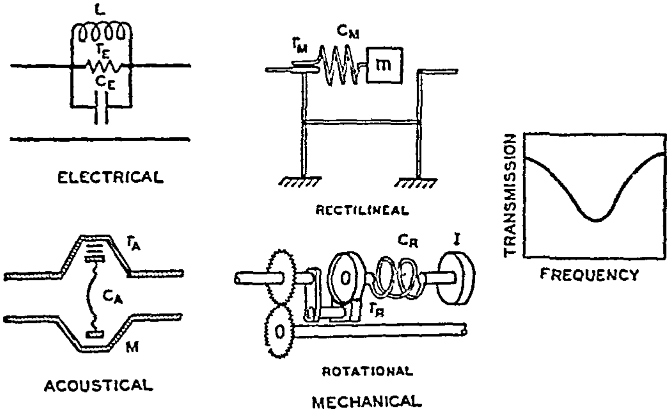

Fig. 5.13. Electrical resistance, inductance and electrical capacitance in parallel, in series with a line and the mechanical rectilineal, mechanical rotational and acoustical analogies. The graph depicts the transmission frequency characteristic.

mechanical rotational and acoustical analogies are also shown in Fig. 5.13. The electrical impedance of the electrical network of Fig. 5.13 is

$$z_{E2} = \frac{j\omega r_E L}{r_E - \omega^2 r_E L C_E + j\omega L} \quad 5.64$$

where r_E = electrical resistance, in abohms,
L = inductance, in abhenries, and
C_E = electrical capacitance, in abfarads.

The output current can be obtained from equations 5.39 and 5.64.

The mechanical rectilineal impedance of the mechanical rectilineal system of Fig. 5.13 is

$$z_{M2} = \frac{j\omega r_M m}{r_M - \omega^2 r_M C_M + j\omega m} \quad 5.65$$

where r_M = mechanical rectilineal resistance, in mechanical ohms,
m = mass, in grams, and
C_M = compliance, in centimeters per dyne.

The output velocity can be obtained from equations 5.40 and 5.65.

The mechanical rotational impedance of the mechanical rotational system of Fig. 5.13 is

$$z_{R2} = \frac{j\omega r_R I}{r_R - \omega^2 r_R I C_R + j\omega I} \tag{5.66}$$

where r_R = mechanical rotational resistance, in rotational ohms,
I = moment of inertia in gram (centimeter)^2, and
C_R = rotational compliance, in radians per dyne per centimeter.

The output rotational velocity can be obtained from equations 5.41 and 5.66.

The acoustical impedance of the acoustical system of Fig. 5.13 is

$$z_{A2} = \frac{j\omega r_A M}{r_A - \omega^2 r_A M C_A + j\omega M} \tag{5.67}$$

where r_A = acoustical resistance, in acoustical ohms,
M = inertance, in grams per (centimeter)^4, and
C_A = acoustical capacitance, in (centimeter)^5 per dyne.

The output volume current can be obtained from equations 5.42 and 5.67.

At low frequencies the systems behave the same as those of Fig. 5.8 and the attenuation is small. At high frequencies the systems behave the same as those of Fig. 5.9 and the attenuation is small. At the resonant frequency of the inductance and electrical capacitance the electrical reactance is infinite and, therefore, from equations 5.39 and 5.64 the attenuation is due to the electrical resistance. At the resonant frequency of the mass and compliance, Fig. 5.13, the mechanical rectilineal reactance is infinite. Therefore, from equations 5.40 and 5.65 the attenuation is due to the mechanical rectilineal resistance. At the resonant frequency of the moment of inertia and the rotational compliance, Fig. 5.13, the mechanical rotational impedance is infinite. Therefore, from equations 5.41 and 5.66 the attenuation is due to the mechanical rotational resistance. At the resonant frequency of the inertance and acoustical capacitance, Fig. 5.13, the acoustical reactance is infinite. Therefore, from

equations 5.42 and 5.67 the attenuation is due to the acoustical resistance. The transmission characteristic of these systems is shown in Fig. 5.13. This characteristic at the low and high frequencies is the same as that of Fig. 5.11. However, in the region of resonance the resistance in each of the four systems decreases the attenuation as depicted by the transmission characteristic of Fig. 5.13.

5.17. Resistance Networks

The use of resistance networks in electrical circuits is well known. Series and shunt networks are employed to introduce dissipation or attenuation in the electrical circuits. In the same way mechanical and acoustical resistance may be used in these systems to introduce dissipation, damping or attenuation. "T" and "π" networks are a combination of series and shunt elements usually employed to introduce attenuation without mismatching impedances or for matching dissimilar impedances.

5.18. Electrical Resistance in Series with a Line and the Mechanical Rectilineal, Mechanical Rotational, and Acoustical Analogies

Fig. 5.14 shows an electrical resistance in series with a line. Referring to equation 5.39 it will be seen that the attenuation will be greater as the

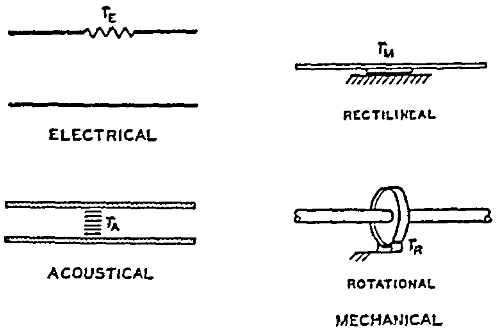

FIG. 5.14. Electrical resistance in series with a line and the mechanical rectilineal, mechanical rotational and acoustical analogies.

resistance is made larger. Equation 5.40 shows that in the same way the attenuation in the mechanical rectilineal system of Fig. 5.14 will be greater as the sliding resistance is made larger. Equation 5.41 shows that the attenuation in the mechanical rotational system of Fig. 5.14 will be

greater as the sliding resistance on the brake wheel is made larger. The acoustical system of Fig. 5.14 shows a system of slits in series with input and output acoustical impedances. Equation 5.42 shows that the attenuation in this system will increase as the acoustic resistance is made larger.

5.19. Electrical Resistance in Shunt with a Line and the Mechanical Rectilineal, Mechanical Rotational and Acoustical Analogies

Fig. 5.15 shows an electrical resistance in shunt with a line. Referring to equation 5.11 it will be seen that the attenuation in this case will be

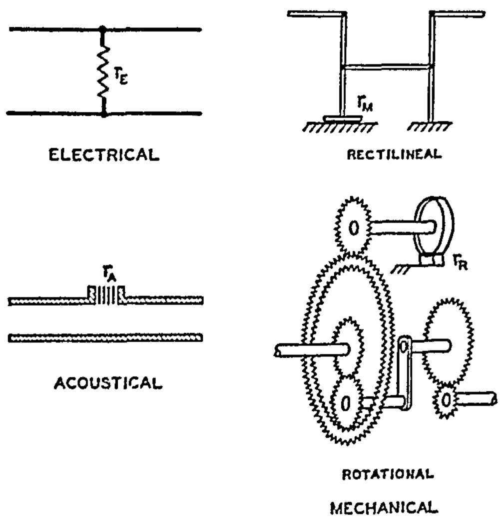

Fig. 5.15. Electrical resistance in shunt with a line and the mechanical rectilineal, mechanical rotational and acoustical analogies.

greater as the electrical resistance is made smaller. Equation 5.12 shows that in the same way the attenuation in the mechanical rectilineal system of Fig. 5.15 will be greater as the sliding resistance is made smaller. Equation 5.13 shows that the attenuation in the mechanical rotational system of Fig. 5.15 will be greater as the sliding resistance on the brake wheel is made smaller. Equation 5.14 shows that in the acoustical system of Fig. 5.15 the attenuation will increase as the shunt acoustical resistance is made smaller.

5.20. "T" Type Electrical Resistance Network and the Mechanical Rectilineal, Mechanical Rotational and Acoustical Analogies

A "T" type electrical network is shown in Fig. 5.16. Equation 5.11 is applicable by considering r_{E1} to be added to z_{E1} and r_{E1} to be added to z_{E3}. r_{E2} is z_{E2} in equation 5.11. In the same way equations 5.12, 5.13 and 5.14 apply to the mechanical rectilineal, mechanical rotational, and

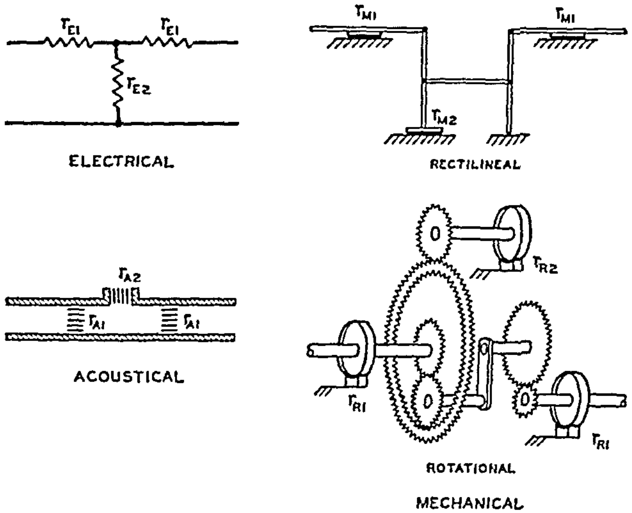

FIG. 5.16. "T" type electrical resistance network and the mechanical rectilineal, mechanical rotational and acoustical analogies.

acoustical systems of Fig. 5.16, wherein z_{M1}, z_{R1}, and z_{A1} is the sum of r_{M1}, r_{R1}, and r_{A1} and the input impedances, respectively, and z_{M3}, z_{R3}, and z_{A3} is the sum of r_{M1}, r_{R1}, and r_{A1} and the output impedances, respectively.

5.21. "π" Type Electrical Resistance Network and the Mechanical Rectilineal, Mechanical Rotational and Acoustical Analogies

A "π" type electrical network is shown in Fig. 5.17. The "π" type of electrical network may be used for the same purpose as the "T" network of the preceding section. The mechanical rectilineal, mechanical rotational, and acoustical resistance systems equivalent to the electrical "π"

network are shown in Fig. 5.17. Equation 5.11 may be used to predict the performance of the electrical system. In this case z_{E1} is the input impedance, z_{E2} is r_{E2}, and z_{E3} is r_{E1} in series with r_{E2} in parallel with the output impedance. This of course determines only the current in z_{E3}. This current is equal to the vector sum of the current in r_{E2} and the

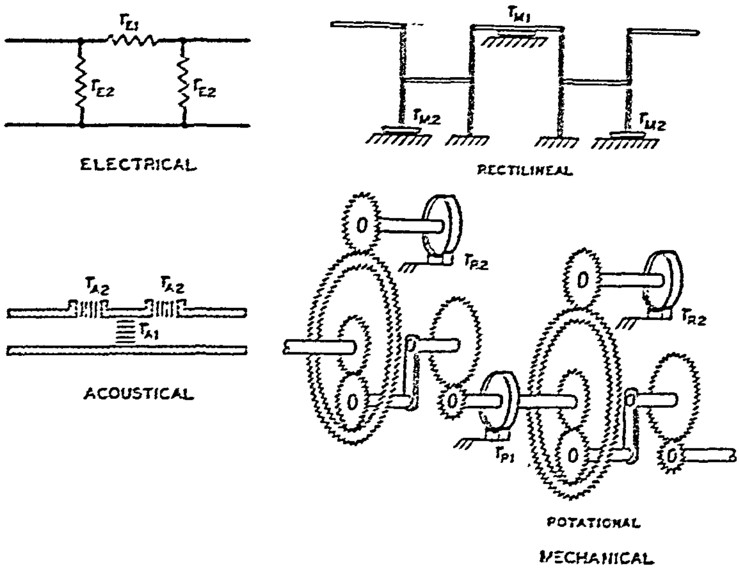

FIG. 5.17. "π" type electrical resistance network and the mechanical rectilineal, mechanical rotational and acoustical analogies.

output impedance. The performance of the mechanical rectilineal, mechanical rotational, and acoustical systems may be obtained by similar considerations employing equations 5.12, 5.13 and 5.14.

5.22. Electrical, Mechanical Rectilineal, Mechanical Rotational and Acoustical Transformers

A transformer is a transducer used for transferring between two impedances of different values without appreciable reflection loss. Electrical, mechanical rectilineal, mechanical rotational and acoustical transformers are shown in Fig. 5.18.

In the ideal electrical transformer of Fig. 5.18 the electromotive force, current and electrical impedance ratios on the two sides of the transformer are

$$e_2 = \frac{N_2}{N_1} e_1 \tag{5.68}$$

$$i_2 = \frac{N_1}{N_2} i_1 \tag{5.69}$$

$$z_{E2} = \left(\frac{N_2}{N_1}\right)^2 z_{E1} \tag{5.70}$$

where N_1 = number of turns in the primary, and
N_2 = number of turns on the secondary.

e_1, i_1, and z_{E1} represent the electromotive force, current and electrical impedance on the primary side and e_2, i_2 and z_{E2} represent the electromotive force current and electrical impedance on the secondary side.

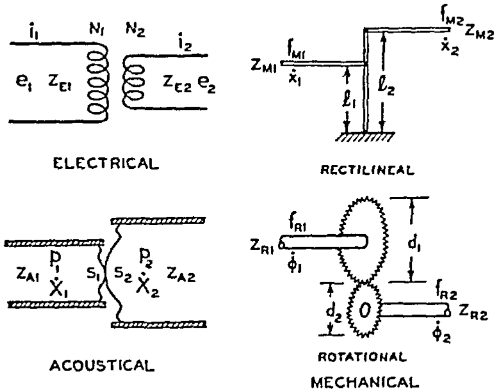

FIG. 5.18. Electrical, mechanical rectilineal, mechanical rotational and acoustical transformers.

The mechanical rectilineal transformer of Fig. 5.18 consists of a rigid massless lever with frictionless bearings. The force, velocity and

mechanical rectilineal impedance ratios on the two sides of the lever are

$$f_{M2} = \frac{l_1}{l_2} f_{M1} \tag{5.71}$$

$$\dot{x}_2 = \frac{l_2}{l_1} \dot{x}_1 \tag{5.72}$$

$$z_{M2} = \left(\frac{l_1}{l_2}\right)^2 z_{M1} \tag{5.73}$$

l_1 and l_2 are the lengths of the lever arms depicted in Fig. 5.18. f_{M1}, $\dot{x}_1$ and z_{M1} and f_{M2}, $\dot{x}_2$ and z_{M2} represents the force, velocity and mechanical rectilineal impedance on the two sides of the mechanical transformer.

The mechanical rotational transformer, of Fig. 5.18, consists of a massless rigid gear train. The torque, angular velocity and mechanical rotational impedance on the two sides of the gear train are

$$f_{R2} = \frac{d_2}{d_1} f_{R1} \tag{5.74}$$

$$\dot{\phi}_2 = \frac{d_1}{d_2} \dot{\phi}_1 \tag{5.75}$$

$$z_{R2} = \left(\frac{d_2}{d_1}\right)^2 z_{R1} \tag{5.76}$$

d_1 and d_2 are the diameters of the gears depicted in Fig. 5.18. f_{R1}, $\dot{\phi}_1$ and z_{R1} and f_{R2}, $\dot{\phi}_2$ and z_{R2} represents the torque, angular velocity and mechanical rotational impedance on the two sides of the rotational transformer.

The acoustical transformer consists of two rigid massless diaphragms with negligible suspension stiffness coupled together as shown in Fig. 5.18. The pressure, volume current and acoustical impedance on the two sides of the diaphragm combination are

$$p_2 = \frac{S_1}{S_2} p_1 \tag{5.77}$$

$$\dot{X}_2 = \frac{S_2}{S_1} \dot{X}_1 \tag{5.78}$$

$$z_{A2} = \left(\frac{S_1}{S_2}\right)^2 z_{A1} \tag{5.79}$$

S_1 and S_2 are the areas of the two diaphragms. p_1, $\dot{X}_1$ and z_{A1} and p_2, $\dot{X}_2$ and z_{A2} represents the pressure, volume current and acoustical impedance on the two sides of the acoustical transformer.

The acoustical transformer of Fig. 5.18 is not purely an acoustical system since it uses mechanical elements in the form of diaphragms. In an acoustical system a horn may be used to transfer from one impedance to another impedance of a different value without appreciable reflection loss. As a matter of fact a horn [6] may be looked upon as an acoustical transformer, transforming large pressures and small volume currents to small pressures and large volume currents or the reverse process.

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[6] For these and other properties of horns see Olson, "Acoustical Engineering," D. Van Nostrand Co., Princeton, N. J., 1957.

CHAPTER VI

WAVE FILTERS

6.1. Introduction

The essential function of a wave filter is to let pass desired frequency bands and to highly attenuate neighboring undesired frequency bands. An electrical filter is a general type of electrical network in which a number of recurrent electrical impedance elements are assembled to form a recurrent structure. Electrical networks of this sort are called *electrical wave filters,*[1] *as they pass certain frequencies freely* and stop others. Wave filters analogous to electrical wave filters may be developed and employed in any wave motion system. Acoustical[2] and mechanical wave filters are becoming very important for use in noise reduction and control of vibrations in all types of vibrating systems. A number of books and numerous articles have been published on electrical wave filters. Therefore it is important to establish the analogies between electrical, mechanical and acoustical wave filters so that the information on electrical wave filters may be used to solve filter problems in mechanical and acoustical systems. It is the purpose of this chapter to illustrate and describe the different types of electrical, mechanical rectilineal, mechanical rotational and acoustical wave filters.

6.2. Types of Wave Filters

The response characteristics of wave filters are widely different. The more frequently used types are designated as follows:

Low Pass Wave Filters
High Pass Wave Filters
Band Pass Wave Filters
Band Elimination Wave Filters

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[1] Campbell, G. A., *Bell System Tech. Jour.*, Vol. I, No. 2, 1922.
[2] Stewart, G. W., *Phys. Rev.*, Vol. 20, No. 6, p. 528, 1922.

A low pass wave filter is a system which passes currents, velocities, angular velocities or volume currents of all frequencies from zero up to a certain frequency termed the cutoff frequency f_C and which bars currents, velocities, angular velocities and volume currents of all higher frequencies.

A high pass wave filter is a system which passes currents, velocities, angular velocities or volume currents of all frequencies from infinity down to a certain frequency termed the cutoff frequency f_C and which bars currents, velocities, angular velocities and volume currents of all lower frequencies.

A band pass wave filter is a system which passes currents, velocities, angular velocities or volume currents that lie between two cutoff frequencies f_{C1} and f_{C2} and bars currents, velocities, angular velocities and volume currents of all frequencies outside this range.

A band elimination wave filter is a system which bars currents, velocities, angular velocities or volume currents that lie between the two cutoff frequencies f_{C1} and f_{C2} and passes currents, velocities, angular velocities and volume currents of all frequencies outside this range.

6.3. Response Characteristics of Wave Filters [3, 4]

The ideal or non-dissipative filters consist entirely of pure reactances. The primary object is the determination of those combinations of reactances which will give a single or double transmitted band of frequencies. The most important type of structure is the ladder type, that is, a certain combination of reactances in series with the line and another combination in shunt with the line. The series reactance and shunt reactance are designated by z_1 and z_2, respectively. It has been shown in treatises on wave filters that attenuation occurs when z_1/z_2 is positive and when z_1/z_2 is negative and no greater in absolute magnitude than four. Non-attenuation occurs when z_1/z_2 is negative and is less in absolute magnitude than four. Therefore, a non-dissipative recurrent structure of the ladder type having series impedances z_1 and shunt impedances z_2 will pass readily only currents of such frequencies as will make the ratio z_1/z_2 lie between 0 and -4.

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[3] Johnson, "Transmission Circuits for Telephonic Communication," D. Van Nostrand Co., Princeton, N. J., 1924.

[4] Shea, "Transmission Networks and Wave Filters," D. Van Nostrand Co., Princeton, N. J., 1929.

6.4. Low Pass Wave Filters

Electrical, mechanical rectilineal, mechanical rotational and acoustical low pass filters are shown in Fig. 6.1.

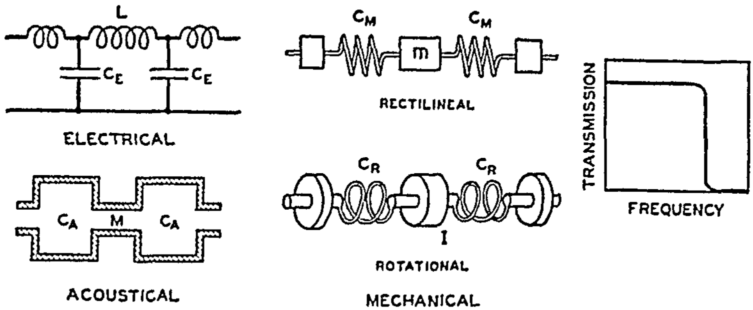

FIG. 6.1. Electrical, mechanical rectilineal, mechanical rotational and acoustical low pass wave filters.

The impedance of the series arm in the four systems is

$$z_{E1} = j\omega L \tag{6.1}$$

$$z_{M1} = j\omega m \tag{6.2}$$

$$z_{R1} = j\omega I \tag{6.3}$$

$$z_{A1} = j\omega M \tag{6.4}$$

The impedance of the shunt arm in the four systems is

$$z_{E2} = \frac{1}{j\omega C_E} \tag{6.5}$$

$$z_{M2} = \frac{1}{j\omega C_M} \tag{6.6}$$

$$z_{R2} = \frac{1}{j\omega C_R} \tag{6.7}$$

$$z_{A2} = \frac{1}{j\omega C_A} \tag{6.8}$$

The limiting frequencies are given by

$$\frac{z_1}{z_2} = 0 \quad \text{and} \quad \frac{z_1}{z_2} = -4 \qquad 6.9$$

From the constants of the systems,

$$\frac{z_{E1}}{z_{E2}} = LC_E\omega_C^2 = 0, \qquad \text{when } \omega_C = 0 \qquad 6.10$$

$$\frac{z_{M1}}{z_{M2}} = mC_M\omega_C^2 = 0, \qquad \text{when } \omega_C = 0 \qquad 6.11$$

$$\frac{z_{R1}}{z_{R2}} = IC_R\omega_C^2 = 0, \qquad \text{when } \omega_C = 0 \qquad 6.12$$

$$\frac{z_{A1}}{z_{A2}} = MC_A\omega_C^2 = 0, \qquad \text{when } \omega_C = 0 \qquad 6.13$$

$$\frac{z_{E1}}{z_{E2}} = -LC_E\omega_C^2 = -4, \quad \text{when } \omega_C = \frac{2}{\sqrt{LC_E}} \qquad 6.14$$

$$\frac{z_{M1}}{z_{M2}} = -mC_M\omega_C^2 = -4, \quad \text{when } \omega_C = \frac{2}{\sqrt{mC_M}} \qquad 6.15$$

$$\frac{z_{R1}}{z_{R2}} = -IC_R\omega_C^2 = -4, \quad \text{when } \omega_C = \frac{2}{\sqrt{IC_R}} \qquad 6.16$$

$$\frac{z_{A1}}{z_{A2}} = -MC_A\omega_C^2 = -4, \quad \text{when } \omega_C = \frac{2}{\sqrt{MC_A}} \qquad 6.17$$

Equations 6.10 and 6.17, inclusive, show that the systems of Fig. 6.1 are low pass filters transmitting currents, linear velocities, angular velocities or volume currents of all frequencies lying between 0 and the cutoff frequency f_C where $f_C = \omega_C/2\pi$.

6.5. High Pass Wave Filters

Electrical, mechanical rectilineal, mechanical rotational, and acoustical high pass wave filters are shown in Fig. 6.2.

The impedance of the series arm in the four systems is

$$z_{E1} = \frac{1}{j\omega C_E} \tag{6.18}$$

$$z_{M1} = \frac{1}{j\omega C_M} \tag{6.19}$$

$$z_{R1} = \frac{1}{j\omega C_R} \tag{6.20}$$

$$z_{A1} = \frac{1}{j\omega C_A} \tag{6.21}$$

For a description of the acoustical capacitance C_A see Sec. 5.12.

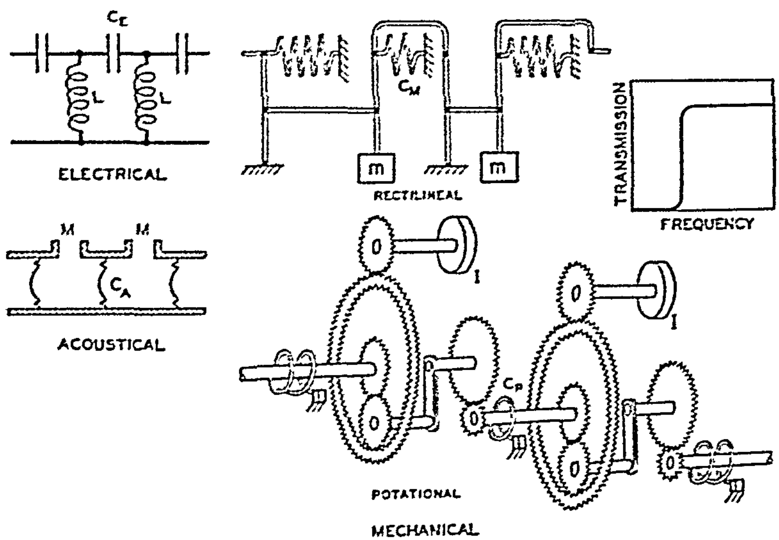

FIG. 6.2. Electrical, mechanical rectilineal, mechanical rotational and acoustical high pass wave filters.

The impedance of the shunt arm in the four systems is

$$z_{E2} = j\omega L \tag{6.22}$$

$$z_{M2} = j\omega m \tag{6.23}$$

$$z_{R2} = j\omega I \tag{6.24}$$

$$z_{A2} = j\omega M \tag{6.25}$$

The limiting frequencies are given by

$$\frac{z_1}{z_2} = 0 \quad \text{and} \quad \frac{z_1}{z_2} = -4 \qquad 6.26$$

From the constants of the systems,

$$\frac{z_{E1}}{z_{E2}} = -\frac{1}{LC_E\omega_C^2} = 0, \qquad \text{when } \omega_C = \infty \qquad 6.27$$

$$\frac{z_{M1}}{z_{M2}} = -\frac{1}{mC_M\omega_C^2} = 0, \qquad \text{when } \omega_C = \infty \qquad 6.28$$

$$\frac{z_{R1}}{z_{R2}} = -\frac{1}{IC_R\omega_C^2} = 0, \qquad \text{when } \omega_C = \infty \qquad 6.29$$

$$\frac{z_{A1}}{z_{A2}} = -\frac{1}{MC_A\omega_C^2} = 0, \qquad \text{when } \omega_C = \infty \qquad 6.30$$

$$\frac{z_{E1}}{z_{E2}} = -\frac{1}{LC_E\omega_C^2} = -4, \quad \text{when } \omega_C = \frac{1}{2\sqrt{LC_E}} \qquad 6.31$$

$$\frac{z_{M1}}{z_{M2}} = -\frac{1}{mC_M\omega_C^2} = -4, \quad \text{when } \omega_C = \frac{1}{2\sqrt{mC_M}} \qquad 6.32$$

$$\frac{z_{R1}}{z_{R2}} = -\frac{1}{IC_R\omega_C^2} = -4, \quad \text{when } \omega_C = \frac{1}{2\sqrt{IC_R}} \qquad 6.33$$

$$\frac{z_{A1}}{z_{A2}} = -\frac{1}{MC_A\omega_C^2} = -4, \quad \text{when } \omega_C = \frac{1}{2\sqrt{MC_A}} \qquad 6.34$$

Equations 6.27 and 6.34, inclusive, show the systems of Fig. 6.2 are high pass wave filters transmitting currents, linear velocities, angular velocities or volume currents of all frequencies lying between the cutoff frequency f_C where $f_C = \omega_C/2\pi$, and infinity.

6.6 Band Pass Wave Filters

Electrical, mechanical rectilineal, mechanical rotational, and acoustical band pass wave filters are shown in Fig. 6.3.

The impedance of the series arm in the four systems is

$$z_{E1} = j\omega L_1 + \frac{1}{j\omega C_{E1}} \tag{6.35}$$

$$z_{M1} = j\omega m_1 + \frac{1}{j\omega C_{M1}} \tag{6.36}$$

$$z_{R1} = j\omega I_1 + \frac{1}{j\omega C_{R1}} \tag{6.37}$$

$$z_{A1} = j\omega M_1 + \frac{1}{j\omega C_{A1}} \tag{6.38}$$

For a description of the acoustical capacitance C_{A1} see Sec. 5.12.

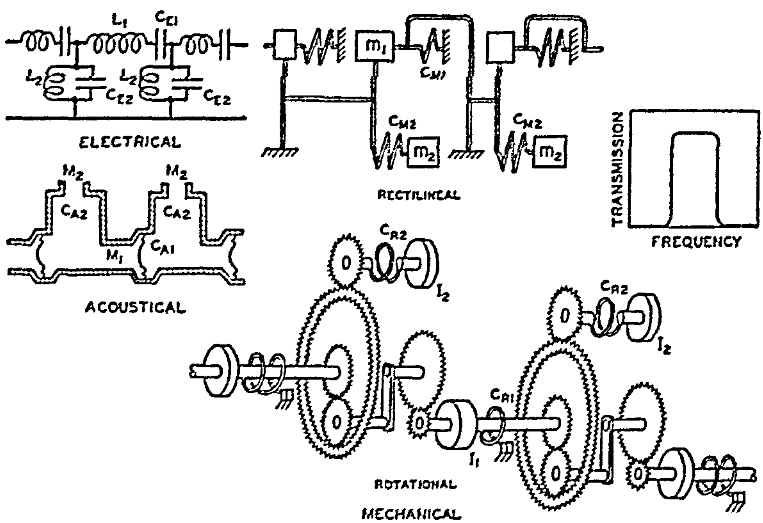

FIG. 6.3. Electrical, mechanical rectilineal, mechanical rotational and acoustical band pass wave filters.

The impedance of the shunt arm in the four systems is

$$z_{E2} = \frac{j\omega L_2}{1 - \omega^2 C_{E2} L_2} \tag{6.39}$$

$$z_{M2} = \frac{j\omega m_2}{1 - \omega^2 C_{M2} m_2} \tag{6.40}$$

$$z_{R2} = \frac{j\omega I_2}{1 - \omega^2 C_{R2} I_2} \tag{6.41}$$

$$z_{A2} = \frac{j\omega M_2}{1 - \omega^2 C_{A2} M_2} \tag{6.42}$$

The limiting frequencies are given by

$$\frac{z_1}{z_2} = 0 \quad \text{and} \quad \frac{z_1}{z_2} = -4 \tag{6.43}$$

Let

$$L_1 C_{E1} = L_2 C_{E2} \tag{6.44}$$

$$m_1 C_{M1} = m_2 C_{M2} \tag{6.45}$$

$$I_1 C_{R1} = I_2 C_{R2} \tag{6.46}$$

$$M_1 C_{A1} = M_2 C_{A2} \tag{6.47}$$

$$\frac{z_{E1}}{z_{E2}} = 0, \quad \text{when } \omega_{C1} = \frac{1}{\sqrt{L_1 C_{E1}}} = \frac{1}{\sqrt{L_2 C_{E2}}} \tag{6.48}$$

$$\frac{z_{M1}}{z_{M2}} = 0, \quad \text{when } \omega_{C1} = \frac{1}{\sqrt{m_1 C_{M1}}} = \frac{1}{\sqrt{m_2 C_{M2}}} \tag{6.49}$$

$$\frac{z_{R1}}{z_{R2}} = 0, \quad \text{when } \omega_{C1} = \frac{1}{\sqrt{I_1 C_{R1}}} = \frac{1}{\sqrt{I_2 C_{R2}}} \tag{6.50}$$

$$\frac{z_{A1}}{z_{A2}} = 0, \quad \text{when } \omega_{C1} = \frac{1}{\sqrt{M_1 C_{A1}}} = \frac{1}{\sqrt{M_2 C_{A2}}} \tag{6.51}$$

$$\frac{z_{E1}}{z_{E2}} = -4, \quad \text{when } \frac{(1 - \omega_{C2}{}^2 L_1 C_{E1})^2}{\omega_{C2}{}^2 L_2 C_{E1}} = 4,$$

or

$$\omega_{C2} = \left[\sqrt{\frac{1}{L_1 C_{E2}} + \frac{1}{L_1 C_{E1}}} \pm \frac{1}{\sqrt{L_1 C_{E2}}}\right] \tag{6.52}$$

$$\frac{z_{M1}}{z_{M2}} = -4, \quad \text{when } \frac{(1 - \omega_{C2}{}^2 m_1 C_{M1})^2}{\omega_{C2}{}^2 m_2 C_{M1}} = 4,$$

or

$$\omega_{C2} = \left[\sqrt{\frac{1}{m_1C_{M2}} + \frac{1}{m_1C_{M1}}} \pm \frac{1}{\sqrt{m_1C_{M2}}}\right] \tag{6.53}$$

$$\frac{z_{R1}}{z_{R2}} = -4, \quad \text{when } \frac{(1 - \omega_{C2}{}^2 I_1 C_{R1})^2}{\omega_{C2}{}^2 I_2 C_{R1}} = 4,$$

or

$$\omega_{C2} = \left[\sqrt{\frac{1}{I_1C_{R2}} + \frac{1}{I_1C_{R1}}} \pm \frac{1}{\sqrt{I_1C_{R2}}}\right] \tag{6.54}$$

$$\frac{z_{A1}}{z_{A2}} = -4, \quad \text{when } \frac{(1 - \omega_{C2}{}^2 M_1 C_{A1})^2}{\omega_{C2}{}^2 M_2 C_{A1}} = 4,$$

or

$$\omega_{C2} = \left[\sqrt{\frac{1}{M_1C_{A2}} + \frac{1}{M_1C_{A1}}} \pm \frac{1}{\sqrt{M_1C_{A2}}}\right] \tag{6.55}$$

It will be noted that ω_{C2} has two values, one greater than ω_{C1} and one less than ω_{C1}. Therefore, the upper and lower cutoff frequencies are given by

$$\omega_{C2}' = \left[\sqrt{\frac{1}{L_1C_{E2}} + \frac{1}{L_1C_{E1}}} - \frac{1}{\sqrt{L_1C_{E2}}}\right], \tag{6.56}$$

and

$$\omega_{C2}'' = \left[\sqrt{\frac{1}{L_1C_{E2}} + \frac{1}{L_1C_{E1}}} + \frac{1}{\sqrt{L_1C_{E2}}}\right] \tag{6.57}$$

$$\omega_{C2}' = \left[\sqrt{\frac{1}{m_1C_{M2}} + \frac{1}{m_1C_{M1}}} - \frac{1}{\sqrt{m_1C_{M2}}}\right], \tag{6.58}$$

and

$$\omega_{C2}'' = \left[\sqrt{\frac{1}{m_1C_{M2}} + \frac{1}{m_1C_{M1}}} + \frac{1}{\sqrt{m_1C_{M2}}}\right] \tag{6.59}$$

$$\omega_{C2}' = \left[\sqrt{\frac{1}{I_1C_{R2}} + \frac{1}{I_1C_{R1}}} - \frac{1}{\sqrt{I_1C_{R2}}}\right], \tag{6.60}$$

and

$$\omega_{C2}'' = \left[\sqrt{\frac{1}{I_1 C_{R2}} + \frac{1}{I_1 C_{R1}}} + \frac{1}{\sqrt{I_1 C_{R2}}}\right] \qquad 6.61$$

$$\omega_{C2}' = \left[\sqrt{\frac{1}{M_1 C_{A2}} + \frac{1}{M_1 C_{A1}}} - \frac{1}{\sqrt{M_1 C_{A2}}}\right], \qquad 6.62$$

and

$$\omega_{C2}'' = \left[\sqrt{\frac{1}{M_1 C_{A2}} + \frac{1}{M_1 C_{A1}}} + \frac{1}{\sqrt{M_1 C_{A2}}}\right] \qquad 6.63$$

Equations 6.56 to 6.63, inclusive, show that the systems of Fig. 6.3 are band pass filters transmitting currents, linear velocities, angular velocities or volume currents of all frequencies lying between two cutoff frequencies f_{C2}' and f_{C2}'', where $f_{C2}' = \omega_{C2}'/2\pi$ and $f_{C2}'' = \omega_{C2}''/2\pi$.

6.7. Band Elimination Wave Filters

Electrical, mechanical rectilineal, mechanical rotational and acoustical band elimination wave filters are shown in Fig. 6.4.

The impedance of the series arm in the four systems is

$$z_{E1} = \frac{j\omega L_1}{1 - \omega^2 C_{E1} L_1} \qquad 6.64$$

$$z_{M1} = \frac{j\omega m_1}{1 - \omega^2 C_{M1} m_1} \qquad 6.65$$

$$z_{R1} = \frac{j\omega I_1}{1 - \omega^2 C_{R1} I_1} \qquad 6.66$$

$$z_{A1} = \frac{j\omega M_1}{1 - \omega^2 C_{A1} M_1} \qquad 6.67$$

The impedance of the shunt arm in the four systems is

$$z_{E2} = j\omega L_2 - \frac{j}{\omega C_{E2}} \qquad 6.68$$

$$z_{M2} = j\omega m_2 - \frac{j}{\omega C_{M2}} \qquad 6.69$$

$$z_{R2} = j\omega I_2 - \frac{j}{\omega C_{R2}} \tag{6.70}$$

$$z_{A2} = j\omega M_2 - \frac{j}{\omega C_{A2}} \tag{6.71}$$

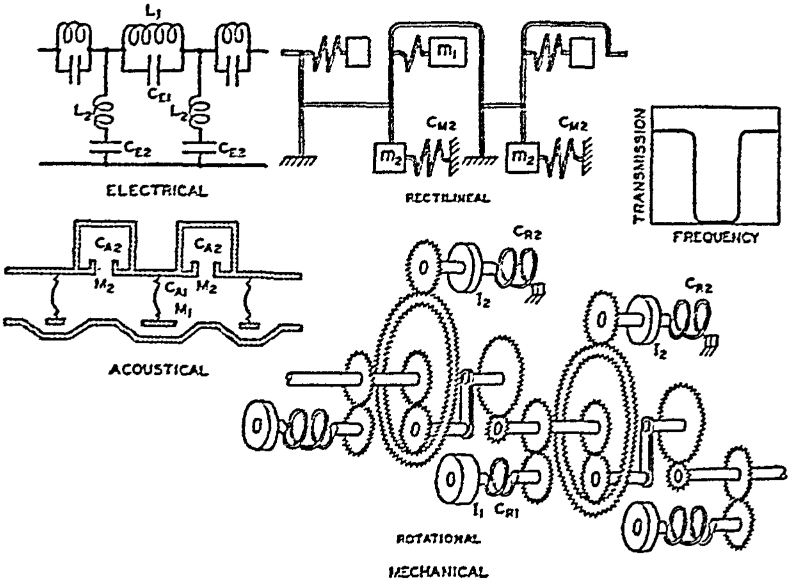

FIG. 6.4. Electrical, mechanical rectilineal, mechanical rotational and acoustical band elimination wave filters.

The limiting frequencies are given by

$$\frac{z_1}{z_2} = 0 \quad \text{and} \quad \frac{z_1}{z_2} = -4 \tag{6.72}$$

Let

$$L_1 C_{E1} = L_2 C_{E2} \tag{6.73}$$

$$m_1 C_{M1} = m_2 C_{M2} \tag{6.74}$$

$$I_1 C_{R1} = I_2 C_{R2} \tag{6.75}$$

$$M_1 C_{A1} = M_2 C_{A2} \tag{6.76}$$

$$\frac{z_{E1}}{z_{E2}} = 0, \quad \text{when } \omega_{C1} = 0 \quad \text{and} \quad \omega_{C4} = \infty \tag{6.77}$$

$$\frac{z_{M1}}{z_{M2}} = 0, \quad \text{when } \omega_{C1} = 0 \quad \text{and} \quad \omega_{C4} = \infty \tag{6.78}$$

$$\frac{z_{R1}}{z_{R2}} = 0, \quad \text{when } \omega_{C1} = 0 \quad \text{and} \quad \omega_{C4} = \infty \tag{6.79}$$

$$\frac{z_{A1}}{z_{A2}} = 0, \quad \text{when } \omega_{C1} = 0 \quad \text{and} \quad \omega_{C4} = \infty \tag{6.80}$$

Two of the limiting frequencies are determined by ω_{C1} and ω_{C4} above.

$$\frac{z_{E1}}{z_{E2}} = -4, \quad \text{when } \frac{L_1 C_{E2}}{(1 - \omega_C{}^2 L_1 C_{E1})^2} = 4,$$

or

$$\omega_C = \frac{\sqrt{L_1 C_{E2} + 16 L_1 C_{E1}} \pm \sqrt{L_1 C_{E2}}}{4 L_1 C_{E1}} \tag{6.81}$$

$$\frac{z_{M1}}{z_{M2}} = -4, \quad \text{when } \frac{m_1 C_{M2}}{(1 - \omega_C{}^2 m_1 C_{M1})} = 4,$$

or

$$\omega_C = \frac{\sqrt{m_1 C_{M2} + 16 m_1 C_M} \pm \sqrt{m_1 C_{M2}}}{4 m_1 C_{M1}} \tag{6.82}$$

$$\frac{z_{R1}}{z_{R2}} = -4, \quad \text{when } \frac{I_1 C_{R2}}{(1 - \omega_C{}^2 I_1 C_{R1})} = 4,$$

or

$$\omega_C = \frac{\sqrt{I_1 C_{R2} + 16 I_1 C_{R1}} \pm \sqrt{I_1 C_{M2}}}{4 I_1 C_{R1}} \tag{6.83}$$

$$\frac{z_{A1}}{z_{A2}} = -4, \quad \text{when } \frac{M_1 C_{A2}}{(1 - \omega_C{}^2 M_1 C_{A1})} = 4,$$

or

$$\omega_C = \frac{\sqrt{M_1 C_{A2} + 16 M_1 C_{A1}} \pm \sqrt{M_1 C_{A2}}}{4 M_1 C_{A1}} \tag{6.84}$$

The other two limiting frequencies are

$$\omega_{C2} = \frac{\sqrt{L_1C_{E2} + 16L_1C_{E1}} - \sqrt{L_1C_{E2}}}{4L_1C_{E1}}, \tag{6.85}$$

and

$$\omega_{C3} = \frac{\sqrt{L_1C_{E2} + 16L_1C_{E1}} + \sqrt{L_1C_{E2}}}{4L_1C_{E1}} \tag{6.86}$$

$$\omega_{C2} = \frac{\sqrt{m_1C_{M2} + 16m_1C_{M1}} - \sqrt{m_1C_{M2}}}{4m_1C_{M1}}, \tag{6.87}$$

and

$$\omega_{C3} = \frac{\sqrt{m_1C_{M2} + 16m_1C_{M1}} + \sqrt{m_1C_{M2}}}{4m_1C_{M1}} \tag{6.88}$$

$$\omega_{C2} = \frac{\sqrt{I_1C_{R2} + 16I_1C_{R1}} - \sqrt{I_1C_{R2}}}{4I_1C_{R1}}, \tag{6.89}$$

and

$$\omega_{C3} = \frac{\sqrt{I_1C_{R2} + 16I_1C_{R1}} + \sqrt{I_1C_{R2}}}{4I_1C_{R1}} \tag{6.90}$$

$$\omega_{C2} = \frac{\sqrt{M_1C_{A2} + 16M_1C_{A1}} - \sqrt{M_1C_{A2}}}{4M_1C_{A1}}, \tag{6.91}$$

and

$$\omega_{C3} = \frac{\sqrt{M_1C_{A2} + 16M_1C_{A1}} + \sqrt{M_1C_{A2}}}{4M_1C_{A1}} \tag{6.92}$$

From equations 6.72 to 6.92, inclusive, it will be seen that the filters of Fig. 6.4 are band elimination filters transmitting currents, linear velocities, angular velocities or volume currents between the frequencies $f_{C1} = 0$ and $f_{C2} = \omega_{C2}/2\pi$, attenuating currents, linear velocities, angular velocities or volume currents between the frequencies $f_{C2} = \omega_{C2}/2\pi$ and $f_{C3} = \omega_{C3}/2\pi$, and transmitting currents, linear velocities, angular velocities or volume currents between the frequencies $f_{C3} = \omega_{C3}/2\pi$ and $f_{C4} = \infty$.

CHAPTER VII

TRANSIENTS

7.1. Introduction

Transients embrace a wide variety of physical phenomena. An electrical transient is the current which flows in a circuit following an electrical disturbance in the system. A mechanical transient is the rectilinear or angular velocity which occurs in a mechanism following a mechanical disturbance in the system. An acoustical transient is the volume current which flows in an acoustical system following an acoustical disturbance in the system. The preceding sections have been concerned with electrical, mechanical and acoustical systems in a steady state condition. The formulas and expressions assume that the systems are in a steady state condition of operation which means that the currents, linear velocities, angular velocities or volume currents have become constant direct or periodic functions of time. The steady state solution is only one part of the solution because immediately after some change in the system the currents or velocities have not settled into a steady state condition. Electrical, mechanical and acoustical systems are subjected to all types of varying and impulsive forces. Therefore, it is important to examine the behavior of these systems when subjected to impulsive forces as contrasted to steady state conditions.

The behavior of a vibrating system may be analyzed by solving the differential equations of the dynamical system. In other words find the currents or velocities of the elements which when substituted in the differential equations will satisfy the initial and final conditions. The solution of the differential equation may be divided [1] into the steady state term and the transient term. The operational calculus is of great value in obtaining the transient response of an electrical, mechanical or acoustical system to a suddenly impressed voltage, force or pressure.

[1] Usually these parts are obtained by solving the differential equation for a particular integral and a complementary function.

The general analysis used by Heaviside is applicable to any type of vibrating system whether electrical, mechanical or acoustical. The response of a system to a unit force can be obtained with the Heaviside calculus. It is the purpose of this section to determine the response of electrical, mechanical rectilineal, mechanical rotational and acoustical systems to a suddenly applied unit electromotive force, force, torque or pressure respectively.

7.2. The Heaviside Operational Calculus [2, 3, 4]

Heaviside's unextended problem is as follows: given a linear dynamical system of n degrees of freedom in a state of equilibrium, find its response when a unit force is applied at any point. The unit function, 1, depicted in Fig. 7.1, is defined to be a force which is zero for $t < 0$ and unity for $t \geq 0$.

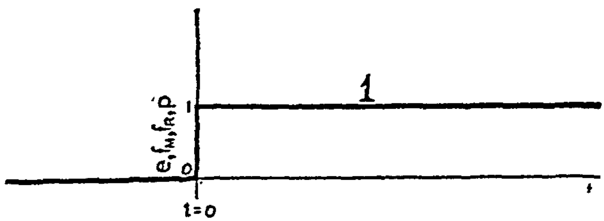

Fig. 7.1. The unit function. The electromotive force, force, torque or pressure is zero before and unity after $t = 0$.

The response of a dynamical system to a unit force is called the indicial admittance of the system. It is denoted by $A(t)$. $A(t)$ represents the current, linear velocity, angular velocity, or volume current when a unit electromotive force, force, torque or pressure is suddenly applied in a system which was initially at rest.

In the Heaviside calculus the differential equations are reduced to an algebraic form by replacing the operator d/dt by the operator p and the operation $\int dt$ by $1/p$. Tables of operational formulas have been compiled which serve for operational calculus the same purpose that tables

[2] Goldman, "Transformation Calculus and Electrical Transients," Prentice-Hall, Inc., New York, N. Y., 1949.

[3] Bush, "Operational Circuit Analysis," John Wiley and Sons, New York, N. Y., 1937.

[4] Berg, "*Heaviside's Operational Calculus*," *McGraw-Hill Book Co.*, New York, N. Y., 1936.

of integrals serve the integral calculus. Operational formulas may be modified, divided or combined by various transformation schemes. This is similar to integration by parts or change of variable in the integral calculus.

The procedure in the Direct Heaviside Operational Method to be followed in obtaining an operational solution of an ordinary differential equation is as follows: Indicate differentiation with respect to the independent variable by means of the operator p. Indicate integration by means of $1/p$. Manipulate p algebraically and solve for the dependent variable in terms of p. Interpret and evaluate the solution in terms of known operators.

7.3. Transient Response of an Inductance and Electrical Resistance in Series and the Mechanical Rectilineal, Mechanical Rotational and Acoustical Analogies

The differential equation of an electromotive force, electrical resistance and inductance connected in series, as shown in Fig. 7.2, is

$$L\frac{di}{dt} + r_E i = e \tag{7.1}$$

where L = inductance, in abhenries,
r_E = electrical resistance, in abohms,
i = current, in abamperes, and
e = electromotive force, in abvolts.

Let the symbol p stand for the operator d/dt, then equation 7.1 becomes

$$Lpi + r_E i = e \tag{7.2}$$

The electrical admittance is

$$\frac{i}{e} = \frac{1}{r_E + Lp} \tag{7.3}$$

If $e = 0$ for $t < 0$ and unity for $t \geq 0$, then the ratio i/e is called the electrical indicial admittance designated as $A_E(t)$. The electrical indicial admittance is

$$A_E(t) = \frac{1}{r_E + Lp}1 \tag{7.4}$$

Equation 7.4 may be written

$$A_E(t) = \frac{1}{(\alpha_E + p)L} 1 \qquad 7.5$$

where $\alpha_E = r_E/L$.

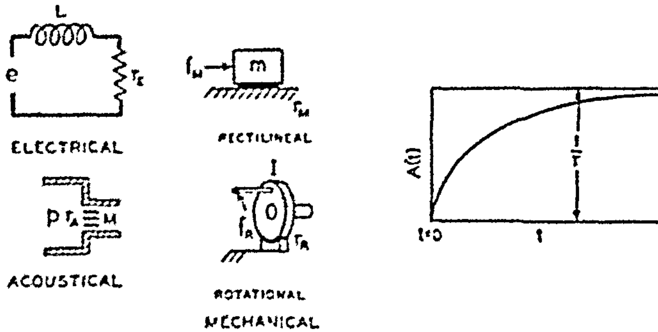

FIG. 7.2. Response of an electrical resistance and inductance in series and the analogous mechanical rectilineal, mechanical rotational and acoustical systems to a unit electromotive force, unit force, unit torque and unit pressure, respectively. The graph depicts the current, velocity, angular velocity or volume current as a function of the time for unit excitation.

From the tables of operational formulas the solution of equation 7.5 is

$$A_E(t) = \frac{1}{L\alpha_E}(1 - \epsilon^{-\alpha_E t}) \qquad 7.6$$

or

$$A_E(t) = \frac{1}{r_E}(1 - \epsilon^{-\frac{r_E}{L}t}) \qquad 7.7$$

The response characteristic is shown in Fig. 7.2. The current is zero for $t = 0$. The current increases for values of $t > 0$ and approaches the value $1/r_E$.

The differential equation of a force driving a mechanical rectilineal resistance and mass, as shown in Fig. 7.2, is

$$m\frac{dv}{dt} + r_M v = f_M \qquad 7.8$$

where m = mass, in grams,

r_M = mechanical rectilineal resistance, in mechanical ohms,

v = linear velocity, in centimeters per second, and

f_M = driving force, in dynes.

The operational equation becomes

$$mpv + r_M v = f_M \tag{7.9}$$

If $f_M = 0$ for $t = 0$ and unity for $t \geq 0$ then the ratio v/f_M is called the mechanical rectilineal indicial admittance designated as $A_M(t)$. The mechanical rectilineal indicial admittance is

$$A_M(t) = \frac{1}{r_M + mp} 1 \tag{7.10}$$

Equation 7.10 may be written

$$A_M(t) = \frac{1}{(\alpha_M + p)m} 1 \tag{7.11}$$

where $\alpha_M = r_M/m$.

From the tables of operational formulas the solution is

$$A_M(t) = \frac{1}{m\alpha_M}(1 - \epsilon^{\alpha_M t}) \tag{7.12}$$

or

$$A_M(t) = \frac{1}{r_M}(1 - \epsilon^{-\frac{r_M}{m}t}) \tag{7.13}$$

The response characteristic is shown in Fig. 7.2. The velocity is zero for $t = 0$. The velocity increases for values of $t > 0$ and approaches the value $1/r_M$.

The differential equation of a torque driving a mechanical rotational resistance and moment of inertia, as shown in Fig. 7.2, is

$$I\frac{d\theta}{dt} + r_R\theta = f_R \tag{7.14}$$

where I = moment of inertia, in gram (centimeter)^2,
r_R = mechanical rotational resistance, in rotational ohms,
θ = angular velocity, in radians per second, and
f_R = torque, in dyne centimeters.

The operational equation becomes

$$Ip\theta + r_R\theta = f_R \tag{7.15}$$

If $f_R = 0$ for $t < 0$ and unity for $t \geq 0$ then the ratio θ/f_R is called the mechanical rotational indicial admittance designated as $A_R(t)$.

The mechanical rotational indicial admittance is

$$A_R(t) = \frac{1}{r_R + Ip} 1 \tag{7.16}$$

Equation 7.16 may be written

$$A_R(t) = \frac{1}{(\alpha_R + p)I} 1 \tag{7.17}$$

where $\alpha_R = r_R/I$.

From the tables of operational formulas, the solution of equation 7.17 is

$$A_R(t) = \frac{1}{I\alpha_R} (1 - \epsilon^{-\alpha_R t}) \tag{7.18}$$

or

$$A_R(t) = \frac{1}{r_R} (1 - \epsilon^{-\frac{r_R}{I}t}) \tag{7.19}$$

The response characteristic is shown in Fig. 7.2. The angular velocity is zero for $t = 0$. The angular velocity increases for values of $t > 0$ and approaches the value $1/r_R$.

The differential equation of a sound pressure driving an acoustical resistance and inertance, as shown in Fig. 7.2, is

$$M\frac{dU}{dt} + r_A U = p \tag{7.20}$$

where M = inertance, in grams per (centimeter)^4,
r_A = acoustical resistance, in acoustical ohms,
U = volume current, in cubic centimeters, and
p = sound pressure, in dynes per square centimeter.

The operational equation becomes

$$MpU + r_A U = p \tag{7.21}$$

If $p = 0$ for $t < 0$ and unity for $t \geq 0$, then the ratio U/p is called the acoustical indicial admittance designated as $A_A(t)$.

The acoustical indicial admittance is

$$A_A(t) = \frac{1}{r_A + Mp} 1 \tag{7.22}$$

Equation 7.22 may be written,

$$A_A(t) = \frac{1}{(\alpha_A + p)M} 1 \qquad 7.23$$

where $\alpha_A = r_A/M$.

From the tables of operational formulas, the solution of equation 7.23 is

$$A_A(t) = \frac{1}{M\alpha_A} (1 - \epsilon^{-\alpha_A t}) \qquad 7.24$$

or

$$A_A(t) = \frac{1}{r_A} (1 - \epsilon^{-\frac{r_A}{M} t}) \qquad 7.25$$

The response characteristic is shown in Fig. 7.2. The volume current is zero for $t = 0$. The volume current increases for values of $t > 0$ and approaches the value $1/r_A$.

7.4. Transient Response of an Electrical Resistance and Electrical Capacitance in Series and the Mechanical Rectilineal, Mechanical Rotational and Acoustical Analogies

The differential equation of an electromotive force, electrical resistance and electrical capacitance connected in series, as shown in Fig. 7.3, is

$$e = r_E i + \frac{1}{C_E} \int i dt \qquad 7.26$$

where C_E = electrical capacitance, in abfarads,
r_E = electrical resistance, in abohms,
i = current, in abamperes, and
e = electromotive force, in abvolts.

The electrical indicial admittance is

$$A_E(t) = \frac{pC_E}{1 + pr_E C_E} 1 = \frac{p}{(\alpha_E + p) r_E} 1 \qquad 7.27$$

where $\alpha_E = 1/r_E C_E$.

From the table of operational formulas, the solution of equation 7.27 is

$$A_E(t) = \frac{\epsilon^{-\alpha_E t}}{r_E} = \frac{\epsilon^{-\frac{t}{r_E C_E}}}{r_E} \qquad 7.28$$

The response characteristic is shown in Fig. 7.3. The current is $1/r_E$ for $t = 0$. The current decreases for values of $t > 0$ and approaches the value zero as a limit.

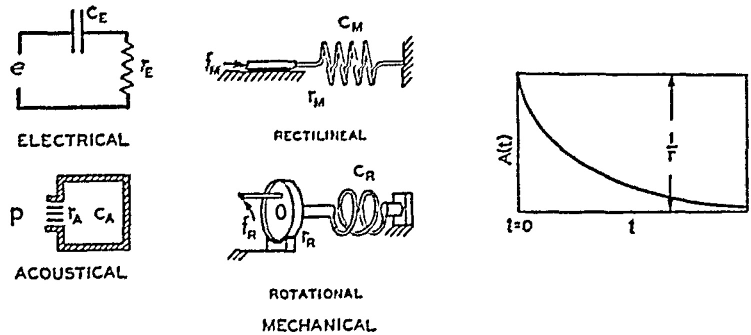

Fig. 7 3. Response of an electrical resistance and electrical capacitance in series and the *analogous mechanical rectilineal, mechanical rotational and acoustical systems to a unit* electromotive force, unit force, unit torque and unit pressure, respectively. The graph depicts the current, velocity, angular velocity or volume current as a function of the time for unit excitation.

The differential equation of a force driving a mechanical rectilineal resistance and compliance, as shown in Fig. 7.3, is

$$f_M = r_M v + \frac{1}{C_M} \int v dt \qquad 7.29$$

where C_M = compliance, in centimeters per dyne,
r_M = mechanical rectilineal resistance, in mechanical ohms,
v = linear velocity, in centimeters, and
f_M = force, in dynes.

The mechanical rectilineal admittance is

$$A_M(t) = \frac{pC_M}{1 + pr_M C_M} 1 = \frac{p}{(\alpha_M + p) r_M} 1 \qquad 7.30$$

where $\alpha_M = 1/r_M C_M$.

From the tables of operational formulas, the solution of equation 7.30 is

$$A_M(t) = \frac{\epsilon^{-\alpha_M t}}{r_M} = \frac{\epsilon^{-\frac{t}{r_M C_M}}}{r_M} \qquad 7.31$$

The response characteristic is shown in Fig. 7.3. The linear velocity is $1/r_M$ for $t = 0$. The velocity decreases for values of $t > 0$ and approaches the value zero as a limit.

The differential equation of a torque driving a mechanical rotational resistance and rotational compliance, as shown in Fig. 7.3, is

$$f_R = r_R\theta + \frac{1}{C_R}\int \theta dt \tag{7.32}$$

where C_R = rotational compliance, in radians per dyne per centimeter,
r_R = mechanical rotational resistance, in rotational ohms,
θ = angular velocity, in radians per second, and
f_R = torque, in dyne centimeters.

The mechanical rotational indicial admittance is

$$A_R(t) = \frac{pC_R}{1 + pr_RC_R}1 = \frac{p}{(\alpha_R + p)r_R}1 \tag{7.33}$$

where $\alpha_R = 1/r_RC_R$.

From the tables of operational formulas the solution of equation 7.33 is

$$A_R(t) = \frac{\epsilon^{-\alpha_R t}}{r_R} = \frac{\epsilon^{-\frac{t}{r_RC_R}}}{r_R} \tag{7.34}$$

The response characteristic is shown in Fig. 7.3. The angular velocity is $1/r_R$ for $t = 0$. The angular velocity decreases for values of $t > 0$ and approaches the value zero as a limit.

The differential equation of a sound pressure driving an acoustical resistance and an acoustical capacitance, as shown in Fig. 7.3, is

$$p = r_AU + \frac{1}{C_A}\int U dt \tag{7.35}$$

where C_A = acoustical capacitance, in (centimeter)^5 per dyne,
r_A = acoustical resistance, in acoustical ohms,
U = volume current, in cubic centimeters per second, and
p = sound pressure, in dynes per square centimeter.

The acoustical indicial admittance is

$$A_A(t) = \frac{pC_A}{1 + pr_AC_A}1 = \frac{p}{(\alpha + p)r_A}1 \qquad 7.36$$

where $\alpha_A = 1/r_AC_A$.

From the tables of operational formulas the solution of equation 7.36 is

$$A_A(t) = \frac{\epsilon^{-\alpha_A t}}{r_A} = \frac{\epsilon^{-\frac{t}{r_AC_A}}}{r_A} \qquad 7.37$$

The response characteristic is shown in Fig. 7.3. The volume current is $1/r_A$ for $t = 0$. The volume current decreases for values of $t > 0$ and approaches the value zero as a limit.

7.5. Transient Response of an Electrical Resistance, Inductance and Electrical Capacitance in Series and the Mechanical Rectilineal, Mechanical Rotational and Acoustical Analogies

The differential equation of an electromotive force, electrical resistance, inductance and electrical capacitance connected in series, as shown in Fig. 7.4, is

$$L\frac{di}{dt} + r_Ei + \frac{1}{C_E}\int idt = e \qquad 7.38$$

where L = inductance, in abhenries,
r_E = electrical resistance, in abohms,
C_E = electrical capacitance, in abohms,
i = current, in abamperes, and
e = electromotive force, in abvolts.

The electrical indicial admittance is

$$A_E(t) = \frac{p}{Lp^2 + r_Ep + \frac{1}{C_E}}1 \qquad 7.39$$

Let

$$\alpha_E = \frac{r_E}{2L}$$

$$\omega_E = \sqrt{\frac{1}{LC_E} - \alpha_E{}^2}$$

The electrical indicial admittance is

$$A_E(t) = \frac{1}{L\omega_E} \frac{p\omega_E}{(p + \alpha_E)^2 + \omega_E^2} 1 \qquad 7.40$$

From the tables of operational formulas, the solution of equation 7.40 is

$$A_E(t) = \frac{1}{L\omega_E} \epsilon^{-\alpha_E t} \sin \omega_E t \qquad 7.41$$

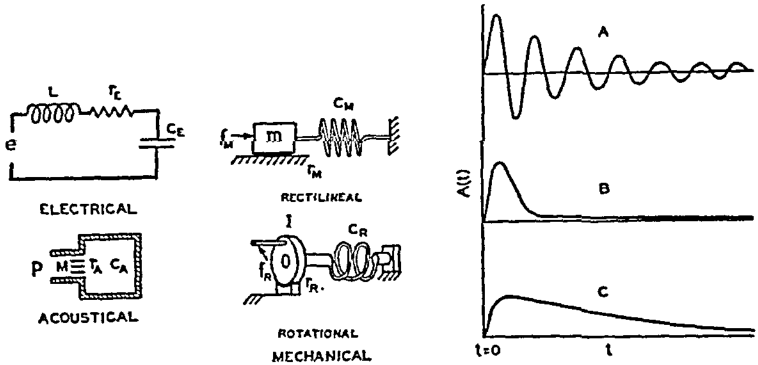

Fig. 7.4. Response of an electrical resistance, inductance and electrical capacitance in series and the analogous mechanical rectilineal, mechanical rotational and acoustical systems to a unit electromotive force, unit force, unit torque or unit pressure, respectively. The graph depicts the current, velocity, angular velocity or volume current as a function of the time for unit excitation.

The response for $r_E^2 < 4L/C_E$ is shown in Fig. 7.4*A*. It is a damped sinusoid.

If $r_E^2 > 4L/C_E$, then the solution becomes

$$A_E(t) = \frac{1}{L\beta_E} \epsilon^{-\alpha_E t} \sinh \beta_E t \qquad 7.42$$

where $\beta_E = \sqrt{\alpha_E^2 - \dfrac{1}{LC_E}}$.

The response for this condition is shown in Fig. 7.4*B*.

If $r_E^2 = 4L/C_E$, then $\sin \omega_E t$ approaches $\omega_E t$ and the solution is

$$A_E(t) = \frac{t}{L}\,\epsilon^{-\alpha_E t} \tag{7.43}$$

The response for this condition is shown in Fig. 7.4*C*.

The differential equation of a force driving a mass, mechanical rectilineal resistance and compliance is shown in Fig. 7.4 as

$$m\ddot{x} + r_M\dot{x} + \frac{x}{C_M} = f_M \tag{7.44}$$

where m = mass, in grams,
r_M = mechanical rectilineal resistance, in mechanical ohms,
C_M = compliance, in centimeters per dyne,
$\ddot{x}$ = acceleration, in centimeters per second per second,
$\dot{x}$ = velocity, in centimeters per second,
x = displacement, in centimeters, and
f_M = driving force, in dynes.

Substituting v for $\dot{x}$, equation 7.44 may be written

$$m\frac{dv}{dt} + r_M v + \frac{1}{C_M}\int v\,dt = f_M \tag{7.45}$$

The mechanical rectilineal indicial admittance is

$$A_M(t) = \frac{p}{mp^2 + r_M p + \dfrac{1}{C_M}}\,1 \tag{7.46}$$

Let

$$\alpha_M = \frac{r_M}{m}$$

$$\omega_M = \sqrt{\frac{1}{mC_M} - \alpha_M^2}$$

The mechanical rectilineal indicial admittance is

$$A_M(t) = \frac{1}{m\omega_M}\,\frac{p\,\omega_M}{(p+\alpha_M)^2 + \omega_M^2}\,1 \tag{7.47}$$

From the tables of operational formulas the solution is

$$A_M(t) = \frac{1}{m\omega_M} \epsilon^{-\alpha_M t} \sin \omega_M t \qquad 7.48$$

The response for $r_M^2 < 4m/C_M$ is shown in Fig. 7.4*A*. It is a damped sinusoid.

If $r_M^2 > 4m/C_M$, then the solution becomes

$$A_M(t) \frac{1}{m\beta_M} \epsilon^{-\alpha_M t} \sinh \beta_M t \qquad 7.49$$

where $\beta_M = \sqrt{\beta_M^2 - \frac{1}{mC_M}}$.

The response of this condition is shown in Fig. 7.4*B*.

If $r_M^2 = 4m/C_M$ then $\sin \omega_M t$ approaches $\omega_M t$ and the solution is

$$A_M(t) = \frac{t}{m} \epsilon^{-\alpha_M t} \qquad 7.50$$

The response for this condition is shown in Fig. 7.4*C*.

The differential equations of a torque driving a moment of inertia, mechanical rotational resistance and rotational compliance, as shown in Fig. 7.4, is

$$I\ddot{\phi} + r_R\dot{\phi} + \frac{\phi}{C_R} = f_R \qquad 7.51$$

where I = moment of inertia, in grams (centimeter),
r_R = mechanical rotational resistance, in rotational ohms,
C_R = rotational compliance, in radians per dyne per centimeter,
$\ddot{\phi}$ = angular acceleration, in radians per second per second,
$\dot{\phi}$ = angular velocity, in radians per second,
ϕ = angular displacement, in radians, and
f_R = driving torque, in dyne centimeters.

Substituting θ for $\dot{\phi}$, equation 7.51 may be written

$$\frac{d\theta}{dt} + r_R\theta + \frac{1}{C_R}\int \theta dt = f_R \qquad 7.52$$

The mechanical rotational indicial admittance is

$$A_R(t) = \frac{p}{Ip^2 + r_R p + \frac{1}{C_R}} 1 \qquad 7.53$$

Let

$$\alpha_R = \frac{r_R}{I}$$

$$\omega_R = \sqrt{\frac{1}{IC_R} + \alpha_R^2}$$

The mechanical rotational indicial admittance is

$$A_R(t) = \frac{1}{I\omega_R} \frac{p\omega_R}{(p + \alpha_R)^2 + \omega_R^2} 1 \qquad 7.54$$

From the tables of operational formulas the solution is

$$A_R(t) = \frac{1}{I\omega_R} \epsilon^{-\alpha_R t} \sin \omega_R t \qquad 7.55$$

The response for $r_R^2 < 4I/C_R$ is shown in Fig. 7.4*A*. It is a damped sinusoid.

If $r_R^2 > 4I/C_R$, then the solution becomes

$$A_R(t) = \frac{1}{I\beta_R} \epsilon^{-\alpha_R t} \sinh \beta_R t \qquad 7.56$$

where $\beta_R = \sqrt{\alpha_R^2 - \frac{1}{IC_R}}$.

The response for this condition is shown in Fig. 7.4*B*.

If $r_R^2 = 4I/C_R$, then $\sin \omega_R t$ approaches $\omega_R t$ and the solution is

$$A_R(t) = \frac{t}{I} \epsilon^{-\alpha_R t} \qquad 7.57$$

The response for this condition is shown in Fig. 7.4*C*.

The differential equation of a sound pressure driving an acoustical resistance and an inertance connected to an acoustical capacitance, as shown in Fig. 7.4, is

$$M\ddot{X} + r_A\dot{X} + \frac{X}{C_A} = p \tag{7.58}$$

where M = inertance, in grams per (centimeter)^4,
r_A = acoustical resistance, in acoustical ohms,
C_A = acoustical capacitance, in (centimeter)^5 per dyne,
$\dot{X}$ = volume current, in cubic centimeters per second, and
p = pressure, in dynes per square centimeter.

Substituting U for $\dot{X}$, equation 7.58 may be written

$$M\frac{dU}{dt} + r_AU + \frac{1}{C_A}\int Udt = p \tag{7.59}$$

The acoustical indicial admittance is

$$A_A(t) = \frac{p}{Mp^2 + r_Ap + \dfrac{1}{C_A}}1 \tag{7.60}$$

Let

$$\alpha_A = \frac{r_A}{2M}$$

$$\omega_A = \sqrt{\frac{1}{MC_A} - {\alpha_A}^2}$$

The acoustical indicial admittance is

$$A_A(t) = \frac{1}{M\omega_A}\frac{p\omega_A}{(p+\alpha_A)^2 + {\omega_A}^2}1 \tag{7.61}$$

From the tables of operational formulas, the solution of equation 7.61 is

$$A_A(t) = \frac{1}{M\omega_A}\epsilon^{-\alpha_At}\sin\omega_At \tag{7.62}$$

The response for ${r_A}^2 < 4M/C_A$ is shown in Fig. 7.4*A*. It is a damped sinusoid.

If $r_A^2 > 4M/C_A$, then the solution becomes

$$A_A(t) = \frac{1}{L\beta_A} \epsilon^{-\alpha_A t} \sinh \beta_A t \qquad 7.63$$

where $\beta_A = \sqrt{\alpha_A^2 - \dfrac{1}{MC_A}}$.

The response for this condition is shown in Fig. 7.4*B*.

If $r_A^2 = 4M/C_A$, then $\sin \omega_A t$ approaches $\omega_A t$ and the solution is

$$A_A(t) = \frac{t}{M} \epsilon^{-\alpha_A t} \qquad 7.64$$

The response for this condition is shown in Fig. 7.4*C*.

7.6. Arbitrary Force

In the preceding sections the response of various combinations of elements to a unit force has been obtained. The value of the unit force

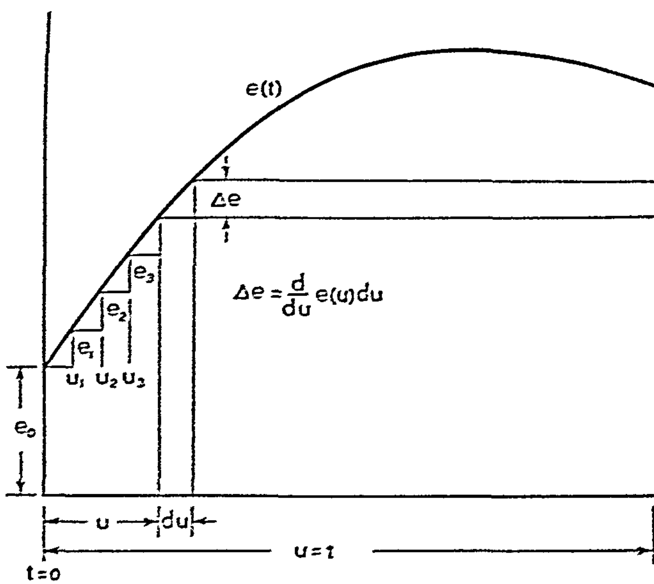

Fig. 7.5. Step function approximation.

solution is that the response to any arbitrary force can be obtained from the unit force solution by a single integration of Duhamel's integral.

It is the purpose of this section to illustrate the proof and use of this integral.

The discussion will be confined to the electrical system. This proof, as in the case of the preceding sections, can be extended to apply to mechanical and acoustical systems. Let the arbitrary electromotive force be represented by the curve of Fig. 7.5. The curve can be assumed to be made up of a series of unit type electromotive forces, as shown in Fig. 7.5. At $t = 0$ an electromotive force e_0 is impressed upon the system. A time u_1 later, an electromotive force e_1 is added, a time u_2 later, an electromotive force e_2 is added, etc., all being of the unit type. The current at a time t is then the sum of the currents due to e_0 at $t = 0$, e_1 at $t = u_1$, etc. The current due to e_0 is $e_0 A_E(t)$, where $A_E(t)$ is the indicial electrical admittance. The current due to the electromotive force Δe, which begins at time u, is obviously $\Delta e A_E(t - u)$, $t - u$ being the time elapsed since the unit electromotive force Δe was turned on. Therefore, the total current at the time $u = t$ is

$$i = e_0 A_E(t) + \sum_{u=0}^{u=t} \Delta e A_E(t - u) \qquad 7.65$$

$$e = \frac{d}{du} e(u) du$$

$$i = e_0 A_E(t) + \int_0^t A_E(t - u) \frac{d}{du} e(u) du \qquad 7.66$$

The above expression may be transformed into different forms. The integral may be transformed by integrating as follows:

$$\int_0^t U dV = UV \Big]_0^t - \int_0^t V dU \qquad 7.67$$

$$U = A(t - u)$$

$$dV = de(u)$$

$$V = e(u)$$

Making the above substitutions, a new expression for the current follows:

$$i = e(t) A_E(0) + \int_0^t e(u) \frac{\partial}{\partial u} A_E(t - u) du \qquad 7.68$$

Equation 7.68 is a fundamental formula which shows the mathematical relation between the current and the type of electromotive force and the constants of the system.

The important conclusions regarding Duhamel's integral are as follows: The indicial admittance of an electrical network determines within a single integration the behavior of a network to any type of electromotive force. In other words a knowledge of the indicial admittance is the only information necessary to completely predict the performance of a system including the steady state.

The velocity, angular velocity and volume current in the mechanical rectilineal, mechanical rotational and acoustical systems are analogous to the equation for the current in the electrical system. Therefore Duhamel's integrals in the mechanical rectilineal, mechanical rotational and acoustical systems are as follows:

$$v = f_M(t)A_M(0) + \int_0^t f_M(u)\frac{\partial}{\partial u}A_M(t-u)du \qquad 7.69$$

$$\theta = f_R(t)A_R(0) + \int_0^t f_R(u)\frac{\partial}{\partial u}A_R(t-u)du \qquad 7.70$$

$$U = p(t)A_A(0) + \int_0^t p(u)\frac{\partial}{\partial u}A_A(t-u)du \qquad 7.71$$

The following general conclusion can be stated as follows: The indicial admittance of any vibrating system determines within a single integration the behavior of the system to any type of applied force.

The Heaviside calculus then becomes an important tool in the solution of transient problems in mechanical and acoustical systems. Since a great number of problems in these fields involve impulsive forces the use of analogies makes it possible to use the tremendous storehouse of information on electrical systems for the solution of problems in mechanical and acoustical systems.

As an example illustrating the use of Duhamel's integral consider an electromotive force $E\epsilon^{-\beta t}$ impressed on the electrical circuit of Fig. 7.3 consisting of an electrical resistance and electrical capacitance in series.

The indicial electrical admittance from equation 7.28 is

$$A_E(t) = \frac{1}{r_E}\epsilon^{-\frac{1}{C_E r_E}t} \qquad 7.72$$

It follows from equation 7.72 that

$$A_E(t-u) = \frac{1}{r_E}\,\epsilon^{-\frac{1}{C_E r_E}(t-u)}$$

$$\frac{\partial}{\partial t} A_E(t-u) = -\frac{1}{C_E {r_E}^2}\,\epsilon^{-\frac{1}{C_E r_E}(t-u)}$$

$$e(u) = E\epsilon^{-\beta u}$$

$$e(t) = E\epsilon^{-\beta t}$$

$$A_E(0) = \frac{1}{r_E}$$

Substituting the above expression in equation 7.68, the current is given by

$$i = E\frac{\epsilon^{-\beta t}}{r_E} + \int_0^t -E\frac{\epsilon^{-\beta u}}{C_E {r_E}^2}\,\epsilon^{-\frac{1}{C_E r_E}(t-u)}\,du \qquad 7.73$$

$$i = \frac{EC_E\cdot}{1 - C_E r_E \beta}\left[\frac{\epsilon^{-\frac{1}{C_E r_E}t}}{C_E r_E} - \beta\epsilon^{-\beta t}\right] \qquad 7.74$$

Similar analysis and analogous equations may be obtained for a combination of a mechanical rectilineal resistance and a compliance, a mechanical rotational resistance and a rotational compliance, and an acoustical resistance and an acoustical capacitance.

CHAPTER VIII

DRIVING SYSTEMS

8.1. Introduction

An electromechanical or electroacoustic transducer or driving system is a system for converting electrical vibrations into the corresponding mechanical or acoustical vibrations. The most common driving systems in use to-day for converting electrical variations into mechanical vibrations are the electrodynamic, the electromagnetic, the electrostatic, the magnetostrictive and the piezoelectric. It is the purpose of this chapter to consider the electrical and mechanical characteristics of these driving systems.

8.2. Electrodynamic Driving System

A moving coil or dynamic driving system is a driving system in which the mechanical forces are developed by the interaction of currents

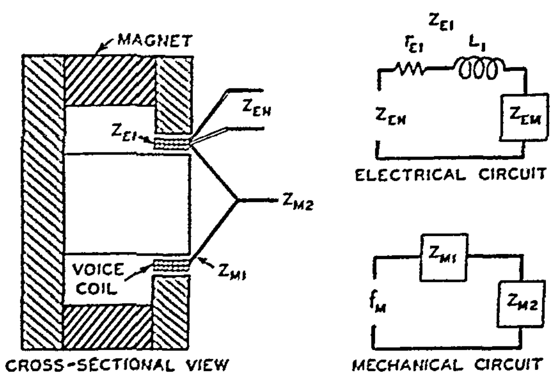

FIG. 8.1. Electrodynamic driving system. In the electrical circuit: z_{EN}, the normal electrical impedance of the voice coil. z_{EM}, the motional electrical impedance of the voice coil. z_{E1}, the damped electrical impedance of the voice coil. $z_{E1} = r_{E1} + j\omega L_1$. L_1 and r_{E1}, the damped inductance and electrical resistance of the voice coil. In the mechanical circuit: f_M, the driving force. z_{M1}, the mechanical rectilineal impedance of the voice coil. z_{M2}, the mechanical rectilineal impedance of the load.

in a conductor and the magnetic field in which it is located. The system is depicted in Fig. 8.1. The force, in dynes, due to the interaction of the current in the voice coil and the polarizing magnetic field is

$$f_M = Bli \tag{8.1}$$

where B = flux density, in gausses,
l = length of the conductor, in centimeters, and
i = current, in abamperes.

The electromotive force, in abvolts, developed by the motion of the conductor is

$$e = Bl\dot{x} \tag{8.2}$$

where $\dot{x}$ = velocity, in centimeters per second.

From equations 8.1 and 8.2

$$\frac{e}{i} = (Bl)^2 \frac{\dot{x}}{f_M} \tag{8.3}$$

$$\frac{e}{i} = z_{EM} \tag{8.4}$$

where z_{EM} = electrical impedance, in abohms, due to motion, termed motional electrical impedance.

From the mechanical circuit [1] of Fig. 8.1, the mechanical rectilineal impedance of the vibrating system at the voice coil is

$$z_M = z_{M1} + z_{M2} \tag{8.5}$$

[1] In the illustrations in the preceding chapters the elements in the electrical network have been labeled r_E, L and C_E. However, in using analogies in actual practice, the conventional procedure is to label the elements in the analogous electrical network with r_M, m and C_M for a mechanical rectilineal system, with r_R, I and C_R for a mechanical rotational system and with r_A, M and C_A for an acoustical system. This procedure will be followed in this chapter in labeling the elements of the analogous electrical circuit. It is customary to label this network with the caption "analogous electrical network of the mechanical system" (or of the rotational system or of the acoustical system) or with the caption "mechanical network" (or rotational network or acoustical network). The latter convention will be used in this chapter. When there is only one path, circuit will be used instead of network.

where z_M = the total mechanical rectilineal impedance at the conductor, in mechanical ohms,

z_{M1} = the mechanical rectilineal impedance of the voice coil and suspension system, in mechanical ohms, and

z_{M2} = the mechanical rectilineal impedance of the load, in mechanical ohms.

The mechanical rectilineal impedance at the voice coil is

$$z_M = \frac{f_M}{\dot{x}} \tag{8.6}$$

The electrical impedance due to motion from equations 8.3, 8.4 and 8.6 is

$$z_{EM} = \frac{(Bl)^2}{z_M} \tag{8.7}$$

The motional electrical impedance of a transducer is the vector difference between its normal and blocked electrical impedance.

The normal electrical impedance of a transducer is the electrical impedance measured at the input to the transducer when the output is connected to its normal load.

The blocked electrical impedance of a transducer is the electrical impedance measured at the input when the mechanical rectilineal system is blocked, that is, in the absence of motion.

The normal electrical impedance z_{EN}, in abohms, of the voice coil is

$$z_{EN} = z_{E1} + z_{EM} \tag{8.8}$$

where z_{E1} = damped electrical impedance of the voice coil, in abohms, and

z_{EM} = motional impedance of the voice coil, in abohms.

The motional electrical impedance as given by equation 8.8 may be represented as in series with the blocked or damped electrical impedance of the conductor, as depicted by the electrical circuit in Fig. 8.1.

The dynamic driving system is almost universally used for all types of direct radiator and horn loud speakers.

8.3. Electromagnetic Driving Systems

A magnetic driving system is a driving system in which the mechanical forces result from magnetic reactions. There are three general types of

magnetic driving systems, namely, the unpolarized armature type, the polarized reed type and the polarized balanced armature type.

A. *Unpolarized Armature Type.*—The unpolarized armature driving system consists of an electromagnet operating directly upon an armature. The armature is spaced at a small distance from the pole piece wound with insulated wire carrying the alternating current. Since there is no polarizing flux, the driving force frequency is twice the frequency of the impressed current to the coil.

Consider the system shown in Fig. 8.2. Assume that all the reluctance

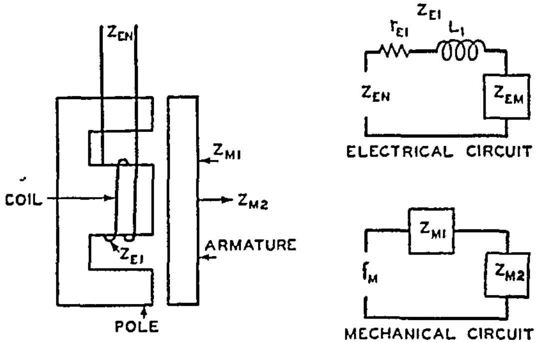

FIG. 8.2. Unpolarized armature electromagnetic driving system. In the electrical circuit: z_{EN}, the normal electrical impedance of the coil. z_{EM}, the motional electrical impedance of the coil. z_{E1}, the damped impedance of the coil. $z_{E1} = r_{E1} + j\omega L_1$. L_1 and r_{E1}, the damped inductance and electrical resistance of the coil. In the mechanical circuit: f_M, the driving force. z_{M1}, the mechanical rectilineal impedance of the armature. z_{M2}, the mechanical rectilineal impedance of the load.

resides in the air gap. The total flux, in maxwells, through the middle pole, in the absence of motion is

$$\phi_T = \frac{2\pi n A i}{a} = \frac{Ci}{a} \tag{8.9}$$

where n = number of turns,

A = area of the middle pole, in square centimeters, it is assumed that the combined area of the two outside poles is equal to the area of the middle pole

a = spacing between the armature and pole, in centimeters,

i = current, in abamperes, and

$C = 2\pi n A$.

Assume that the armature is displaced from its normal position a distance Δx centimeters towards the pole; the total flux is

$$\phi_T + \Delta\phi_T = \frac{Ci}{a - \Delta x} \qquad 8.10$$

Now let the armature be displaced a distance Δx centimeters away from the pole; the total flux is

$$\phi_T - \Delta\phi_T = \frac{Ci}{a + \Delta x} \qquad 8.11$$

The difference in flux for these two conditions is

$$2\Delta\phi_T = \frac{2Ci\Delta x}{a^2 - (\Delta x)^2} \qquad 8.12$$

The change in flux with respect to time is

$$\frac{\Delta\phi_T}{\Delta t} = \frac{Ci}{a^2 - (\Delta x)^2}\frac{\Delta x}{\Delta t} \qquad 8.13$$

The voltage, in abvolts, generated due to motion is

$$e = n\frac{d\phi}{dt} \qquad 8.14$$

If Δx is small compared to a, then from equations 8.13 and 8.14

$$e = \frac{nCi}{a^2}\dot{x} \qquad 8.15$$

The force on the armature, in dynes, is

$$f_M = \frac{(2\phi_T)^2}{16\pi A} = \frac{C^2 i^2}{4\pi A a^2} \qquad 8.16$$

where i = current in the coil, in abamperes,
$C = 2\pi n A$,
A = area of the center pole, in square centimeters,
n = number of turns, and
a = normal spacing, in centimeters.

If the current in the coil is sinusoidal, then the expression for the current can be written

$$i = i_{max} \sin \omega t \tag{8.17}$$

where i_{max} = amplitude of the current in abamperes,
$\omega = 2\pi f$
f = frequency in cycles per second, and
t = time, in seconds.

Substituting equation 8.17 for the current in 8.16, the force on the armature is

$$f_M = \frac{C^2}{4\pi A a^2} i^2_{max} \sin^2 \omega t$$

$$= \frac{C^2}{4\pi A a^2} i^2_{max} (\tfrac{1}{2} - \tfrac{1}{2} \cos 2\omega t) \tag{8.18}$$

Equation 8.18 shows that there is a steady force and an alternating driving force of twice the frequency of the impressed current.

From the mechanical circuit of Fig. 8.2, the mechanical rectilineal impedance of the vibrating system is

$$z_M = z_{M1} + z_{M2} \tag{8.19}$$

where z_M = total mechanical rectilineal impedance at the armature, in mechanical ohms,
z_{M1} = mechanical rectilineal impedance of the armature, in mechanical ohms, and
z_{M2} = mechanical rectilineal impedance of the load, in mechanical ohms.

The mechanical rectilineal impedance at the armature is

$$z_M = \frac{f_M}{\dot{x}} \tag{8.20}$$

From equations 8.15 and 8.16

$$\frac{e}{i} = \frac{nC^3 i^2}{4\pi A a^4} \frac{\dot{x}}{f_M} \tag{8.21}$$

From equations 8.19, 8.20 and 8.21

$$z_{EM} = \frac{2\pi^2 n^4 A^2 i^2}{a^4 z_M} \qquad 8.22$$

where z_{EM} = motional electrical impedance, in abohms,
n = number of turns,
A = area of center pole, in square centimeters,
i = current, in abamperes,
a = spacing, in centimeters, and
z_M = mechanical rectilineal impedance of the load including the armature, in mechanical ohms.

The normal electrical impedance z_{EN}, in abohms, of the coil is

$$z_{EN} = z_{E1} + z_{EM} \qquad 8.23$$

where z_{E1} = damped electrical impedance of the coil, in abohms, and
z_{EM} = motional electrical impedance of the coil, in abohms.

The motional electrical impedance as given by equation 8.23 may be represented as in series with the blocked or damped electrical impedance of the coil as depicted by the electrical circuit in Fig. 8.2.

The frequency of vibration of the armature is twice the frequency of the impressed electrical current. Therefore, this system cannot be used for the reproduction of sound. It is, however, a simple driving system for converting electrical variations into mechanical vibrations of double frequency. The unpolarized driving system is used for low frequency supersonic generators, saws, filing machines, vibrators and clippers.

B. *Polarized Reed Armature Type.*—A reed armature driving system consists of an electromagnet operating directly upon an armature of steel as in Fig. 8.3. The steel armature is spaced at a small distance from a pole piece wound with insulated wire carrying the alternating current and supplied with steady flux from the poles of a permanent magnet.

The flux, in maxwells, due to the permanent magnet is given by

$$\phi_1 = \frac{M}{R_1} \qquad 8.24$$

where M = magnetomotive force of the magnet, in gilberts, and
R_1 = reluctance of the magnetic circuit, in oersteds.

The flux, in maxwells, due to the sinusoidal current $i_{\max} \sin \omega t$ in the coils is given by

$$\phi_2 = \frac{4\pi N i_{\max} \sin \omega t}{R_2} \tag{8.25}$$

where N = number of turns in the coil,
i = current in the coil, in abamperes,
R_2 = reluctance of the alternating magnetic circuit, in oersteds,
$\omega = 2\pi f$,
f = frequency, and
t = time.

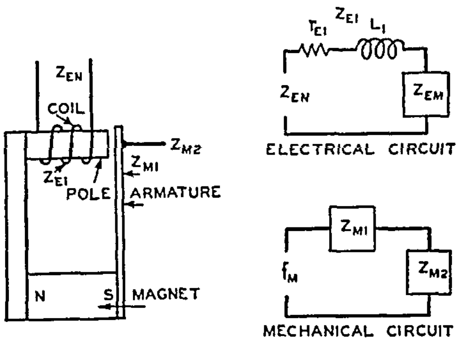

FIG. 8.3. Polarized reed armature electromagnetic driving system. In the electrical circuit: z_{EN}, the normal electrical impedance of the coil. z_{EM}, the motional electrical impedance of the coil. z_{E1}, the damped impedance of the coil. $z_{E1} = r_{E1} + j\omega L_1$. L_1 and r_{E1}, the damped inductance and electrical resistance of the coil. In the mechanical circuit: f_M, the driving force. z_{M1}, the mechanical rectilineal impedance of the armature system. z_{M2}, the mechanical rectilineal impedance of the load.

The force, in dynes, on the armature is

$$f_M = \frac{(\phi_1 + \phi_2)^2}{8\pi A} = \frac{M^2}{8\pi R_1{}^2 A} + \frac{MNi_{\max} \sin \omega t}{R_1 R_2 A} + \frac{\pi N^2 i^2{}_{\max}}{R_2{}^2 A} - \frac{\pi N^2 i^2{}_{\max} \cos 2\omega t}{R_2{}^2 A} \tag{8.26}$$

where A = effective area of the pole in square centimeters.

The first and third terms of the right hand side of equation 8.26 represent a steady force, the second term represents a force of the same fre-

quency as the alternating current and the last term represents a force of twice the frequency of the alternating current. Referring to equation 8.26 it will be seen that the driving force is proportional to the steady flux ϕ_1. Also ϕ_1 must be large compared to ϕ_2, in order to reduce second harmonic distortion. For these reasons the polarizing flux should be made as large as possible.

The motional electrical impedance of this system will now be considered. If all the reluctance is assumed to reside in the air gap, the flux, in maxwells, through the armature is

$$\phi_1 = \frac{MA}{a} \tag{8.27}$$

where M = magnetomotive force, in gilberts, due to the steady field,
a = spacing between the armature and pole, in centimeters, and
A = area of the pole, in square centimeters.

Let the armature be deflected a distance Δx towards the pole; the flux will now be

$$\phi_1 + \Delta\phi_1 = \frac{MA}{a - \Delta x} \tag{8.28}$$

Now let the armature be pulled away from the normal position a distance of Δx; the flux will be

$$\phi_1 - \Delta\phi_1 = \frac{MA}{a + \Delta x} \tag{8.29}$$

The difference in flux through the armature for these two conditions is

$$2\Delta\phi_1 = \frac{MA}{a - \Delta x} - \frac{MA}{a + \Delta x} = \frac{2MA\Delta x}{a^2 - (\Delta x)^2} \doteq \frac{2MA\Delta x}{a^2} \tag{8.30}$$

This change in flux with respect to the time is

$$\frac{d\phi_1}{dt} = \frac{MA}{a^2}\frac{dx}{dt} \tag{8.31}$$

The electromotive force, in abvolts, generated in the coil due to this deflection of the armature is

$$e = N\frac{d\phi_1}{dt} = \frac{NMA}{a^2}\dot{x} \tag{8.32}$$

Leaving out the steady force and the force of twice the frequency, equation 8.26 becomes

$$f_M = \frac{MNi}{R_1 R_2 A} \qquad 8.33$$

From the mechanical circuit of Fig. 8.3, the mechanical rectilineal impedance of the vibrating system is

$$z_M = z_{M1} + z_{M2} \qquad 8.34$$

where z_M = total mechanical rectilineal impedance at the armature directly above the pole piece, in mechanical ohms,

z_{M1} = mechanical rectilineal impedance of the armature, in mechanical ohms, and

z_{M2} = mechanical rectilineal impedance of the load in mechanical ohms.

The mechanical rectilineal impedances z_M, z_{M1}, and z_{M2} are referred to a point on the armature directly over the pole piece.

The mechanical rectilineal impedance of the armature directly above the pole piece is

$$z_M = \frac{f_M}{\dot{x}} \qquad 8.35$$

Combining equations 8.32 and 8.33,

$$\frac{e}{i} = \frac{\dot{x}}{f_M} \frac{M^2 N^2}{R_1 R_2 a^2} \qquad 8.36$$

From equations 8.34, 8.35 and 8.36

$$z_{EM} = \frac{M^2 N^2}{R_1 R_2 a^2 z_M} \qquad 8.37$$

where z_{EM} = motional impedance, in abohms,

z_M = total mechanical impedance with reference to a point on the armature directly over the pole piece.

From equations 8.24 and 8.37, assuming $R_1 = R_2$

$$z_{EM} = \frac{\phi_1^2 N^2}{a^2 z_M} \qquad 8.38$$

Equation 8.38 is similar to equation 8.7 for the electrodynamic system. The normal electrical impedance z_{EN}, in abohms, of the coil is

$$z_{EN} = z_{E1} + z_{EM} \tag{8.39}$$

where z_{E1} = damped electrical impedance of the coil, in abohms, and
z_{EM} = motional electrical impedance of the coil, in abohms.

The motional electrical impedance as given by equation 8.38 may be represented as in series with the blocked or damped electrical impedance of the coil as depicted by the electrical circuit in Fig. 8.3.

This driving system is not generally used in loud speakers. The most common example of this driving system is the bipolar telephone receiver where the diaphragm is the armature.

C. *Polarized Balanced Armature Type.*—There are innumerable possibilities in the design of a magnetic driving system. The preceding

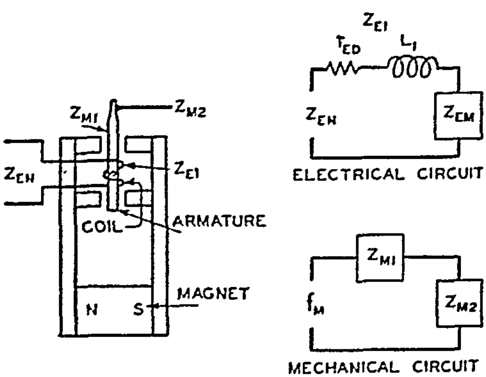

Fig. 8.4. Polarized balanced armature electromagnetic driving system. In the electrical circuit: z_{EN}, the normal electrical impedance of the coil. z_{EM}, the motional electrical impedance of the coil. z_{E1}, the damped electrical impedance of the coil. $z_{E1} = r_{E1} + j\omega L_1$. L_1 and r_{E1}, the damped inductance and electrical resistance of the coil. In the mechanical circuit: f_M, the driving force. z_{M1}, the mechanical rectilineal impedance of the armature. z_{M2}, the mechanical rectilineal impedance of the load. z_{M1} and z_{M2} are referred to a point on the armature directly over a pole piece.

section considered the simplest magnetic driving system in which both the steady flux and the alternating flux flow through the armature. It is the purpose of this section to consider the balanced armature type of driving system in which only the alternating flux flows through the armature.

A typical balanced armature driving system is shown in Fig. 8.4. The steady field is usually supplied by a permanent magnet. The armature is located so that it is in the equilibrium with the steady forces. The alternating current winding is wound around the armature. The steady force, in dynes, at the poles (Fig. 8.4) due to the magnetic field is

$$f_M = \frac{\phi_1^2}{8\pi A} \tag{8.40}$$

where ϕ_1 = total flux, in maxwells, at each pole due to the permanent magnet, and
A = effective area, in square centimeters, of the pole piece.

The flux, in maxwells, at the poles due to a current in the coil is

$$\phi_2 = \frac{4\pi Ni}{R_2} \tag{8.41}$$

where N = number of turns in the coil,
i = current in the coil, in abamperes, and
R_2 = reluctance of the magnetic circuit, in oersteds, which the coil energizes.

The sum of the forces, in dynes, at the four poles acting upon the armature due to a current in the coil is

$$f_M = \frac{2(\phi_1 + \phi_2)^2}{8\pi A} - \frac{2(\phi_1 - \phi_2)^2}{8\pi A} = \frac{\phi_1\phi_2}{\pi A} \tag{8.42}$$

or

$$f_M = \frac{4\phi_1 Ni}{R_2 A} \tag{8.43}$$

In the case of the simple reed driving system a second harmonic term appeared in the force when a sinusoidal current was passed through the coil. It is interesting to note that in the case of the balanced armature the second harmonic term cancels out due to the push pull arrangement.

The motional impedance of this system will now be considered. Let the armature be deflected clockwise a distance of Δx from the poles. The

flux, in maxwells, through the armature to the right and upward, assuming that the entire reluctance exists in the air gap, is

$$\phi_1 + \Delta\phi_1 = \frac{MA}{2(a - \Delta x)} \tag{8.44}$$

where M = magnetomotive force, in gilberts, of the steady field,
a = spacing between the armature and pole, in centimeters, and
A = effective area of a pole piece, in square centimeters.

The flux through the armature to the left and downward is

$$\phi_1 - \Delta\phi_1 = \frac{MA}{2(a + \Delta x)} \tag{8.45}$$

The flux through the armature is the difference between equations 8.44 and 8.45.

$$\Delta\phi_1 = \frac{MA\Delta x}{a^2 - (\Delta x)^2} \doteq \frac{MA\Delta x}{a^2} \tag{8.46}$$

The change in flux with respect to the time is

$$\frac{d\phi}{dt} = \frac{M}{a^2}\frac{dx}{dt} = \frac{MA}{a^2}\dot{x} \tag{8.47}$$

The electromotive force, in abvolts, generated in the coil is

$$e = N\frac{d\phi}{dt} = \frac{NMA}{a^2}\dot{x} \tag{8.48}$$

From the mechanical circuit of Fig. 8.4, the mechanical rectilineal impedance of the vibrating system is

$$z_M = z_{M1} + z_{M2} \tag{8.49}$$

where z_M = total mechanical rectilineal impedance, in mechanical ohms,
z_{M1} = mechanical rectilineal impedance of the armature, in mechanical ohms, and
z_{M2} = mechanical rectilineal impedance of the load, in mechanical ohms.

The mechanical rectilineal impedances z_M, z_{M1}, and z_{M2} are referred to a point on the armature directly above a pole piece.

The mechanical rectilineal impedance at the armature directly over a pole piece is

$$z_M = \frac{f_M}{\dot{x}} \qquad 8.50$$

Combining equations 8.43 and 8.50,

$$\frac{e}{i} = \frac{4N^2\phi_1 M}{a^2 R_2} \frac{\dot{x}}{f_M} \qquad 8.51$$

From equations 8.49, 8.50 and 8.51

$$z_{EM} = \frac{4N^2\phi_1 M}{a^2 R_2 z_M} \qquad 8.52$$

where z_{EM} = motional electrical impedance, in abohms, and
z_M = total mechanical rectilineal impedance including the armature with reference to a point on the armature directly over one of the pole pieces.

If the entire reluctance is assumed to reside in the air gap, equation 8.52 may be written

$$z_{EM} = \frac{4N^2\phi_1{}^2}{a^2 z_M} \qquad 8.53$$

Equation 8.53 is essentially the same as equation 8.38 for the reed armature type and is similar to equation 8.7 for the electrodynamic system. The normal electrical impedance z_{EN}, in abohms, of the coil is

$$z_{EN} = z_{E1} + z_{EM} \qquad 8.54$$

where z_{E1} = damped electrical impedance of the coil, in abohms, and
z_{EM} = motional electrical impedance of the coil, in abohms.

The motional electrical impedance as given by equation 8.53 may be represented as in series with the blocked or damped electrical impedance of the coil as depicted by the electrical circuit in the Fig. 8.4.

When the armature is displaced by the current, means must be provided for returning the armature to the equilibrium position. Due to

Assume that the polarizing voltage is e_0 and that the alternating voltage is $e = e_{max} \sin \omega t$. The force, in dynes, between the plates is

$$f_M = \frac{(e_0 + e_{max} \sin \omega t)^2 A}{8\pi a^2} \tag{8.56}$$

$$f_M = \frac{e_0{}^2 + 2e_0 e_{max} \sin \omega t + \frac{1}{2}e^2{}_{max} - \frac{1}{2}e^2{}_{max} \cos 2\omega t}{8\pi a^2} A \tag{8.57}$$

The first and third terms in the numerator of equation 8.57 represent steady forces. The fourth term is an alternating force of twice the frequency of the impressed voltage. The second term is an alternating force of the frequency of the impressed voltage. If the polarizing electromotive e_0 is large compared to the alternating electromotive force $e_{max} \sin \omega t$, the fourth term will be negligible. The useful force, in dynes, then is the second term which causes the moving surface to vibrate with a velocity which corresponds to the impressed electromotive force.

$$f_M = \frac{e_0 e_{max} \sin \omega t}{4\pi a^2} A = \frac{e_0 e}{4\pi a^2} A \tag{8.58}$$

The motional impedance of this system will now be considered. The charge, in statcoulombs, on the condenser is

$$q = C_E e_0 \tag{8.59}$$

where e_0 = potential difference between the plates, in statvolts, and
C_E = capacity per unit area, in statfarads.

The current, in statamperes, generated due to motion is

$$i = \frac{dq}{dt} \tag{8.60}$$

From equations 8.59 and 8.60 the generated current is

$$i = e_0 \frac{dC_E}{dx} \frac{dx}{dt} \tag{8.61}$$

The capacitance of the condenser, in statfarads, is

$$C_{E1} = \frac{A}{4\pi a} \tag{8.62}$$

Let the movable plate be deflected a distance Δx away from the fixed plate. The capacitance is

$$C_{E1} - \Delta C_{E1} = \frac{A}{4\pi(a + \Delta x)} \qquad 8.63$$

Now let the movable plate be deflected a distance Δx towards the fixed plate. The capacitance is

$$C_{E1} + \Delta C_{E1} = \frac{A}{4\pi(a - \Delta x)} \qquad 8.64$$

The difference between the two conditions is

$$\Delta C_{E1} = \frac{A\Delta x}{4\pi[a^2 - (\Delta x)^2]} \doteq \frac{A\Delta x}{4\pi a^2} \qquad 8.65$$

The change in capacitance with respect to x is

$$\frac{dC_{E1}}{dx} = \frac{1}{4\pi a^2} \qquad 8.66$$

Substituting equation 8.66 in 8.61, the generated current, in statamperes, is

$$i = \frac{e_0 A}{4\pi a^2}\dot{x} \qquad 8.67$$

From the mechanical circuit of Fig. 8.5, the mechanical rectilineal impedance of the vibrating system is

$$z_M = z_{M1} + z_{M2} \qquad 8.68$$

where z_M = total mechanical rectilineal impedance of the vibrating system, in mechanical ohms,

z_{M1} = mechanical rectilineal impedance of the vibrating plate, in mechanical ohms, and

z_{M2} = mechanical rectilineal impedance of the load, in mechanical ohms.

The mechanical rectilineal impedance at the plate is

$$z_M = \frac{f_M}{\dot{x}} \qquad 8.69$$

From equations 8.58 and 8.67

$$\frac{e}{i} = \frac{16\pi^2 a^4}{e_0{}^2 A^2} \frac{f_M}{\dot{x}} \qquad 8.70$$

From equations 8.68, 8.69 and 8.70

$$z_{EM} = \frac{16\pi^2 a^4}{e_0{}^2 A^2} z_M \qquad 8.71$$

where z_{EM} = motional electrical impedance, in statohms, and
z_M = total mechanical rectilineal impedance presented to the vibrating surface including the vibrating surface.

The normal electrical impedance z_{EN}, in statohms, of the condenser is

$$z_{EN} = \frac{z_{E1} z_{EM}}{z_{E1} + z_{EM}} \qquad 8.72$$

where z_{E1} = damped electrical impedance of the condenser, in statohms, and
z_{EM} = motional electrical impedance of the condenser, in statohms.

The motional electrical impedance as given by equation 8.72 may be represented as in parallel with the blocked or damped electrical impedance of the condenser as depicted by the electrical network in Fig. 8.5.

The condenser driving system has been employed as a loud speaker in which case the moving electrode radiates directly into the air. Means must be provided to keep the electrodes separated without, at the same time, adding a large stiffness. In a bilateral or push pull arrangement the movable electrode is placed between two stationary plates and the large steady forces are balanced out.

8.5. Magnetostriction Driving System

A magnetostriction driving system is a driving system in which the mechanical forces result from the deformation of a ferromagnetic mate-

rial having magnetostriction properties. The term "Joule effect" is applied to the phenomena in which a change in linear dimensions occurs when a magnetic field is applied along a specified direction. The term "Villari effect" is applied to the phenomena in which a change in magnetic induction occurs when a mechanical stress is applied along a specified direction.

Consider the system shown in Fig. 8.6. Assume that the rod is clamped so that no motion is possible and that a current is applied to

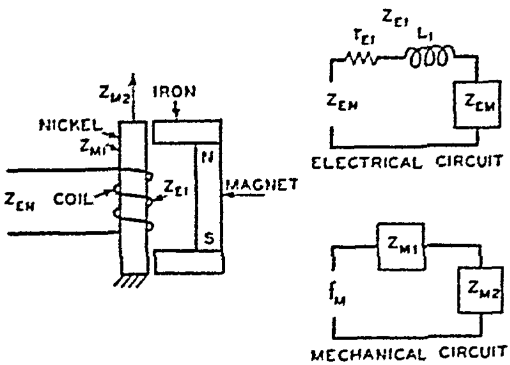

Fig. 8.6. Magnetostriction driving system. In the electrical circuit: z_{EN}, the normal electrical impedance of the coil. z_{EM}, the motional electrical impedance of the coil. z_{E1}, the damped electrical impedance of the coil. $z_{E1} = r_{E1} + j\omega L_1$. L_1 and r_{E1}, the damped inductance and electrical resistance of the coil. In the mechanical circuit: f_M, the driving force. z_{M1}, the mechanical rectilineal impedance of the rod. z_{M2}, the mechanical rectilineal impedance of the load.

the winding; then the mechanical stress, in dynes, due to the Joule effect, is

$$f_M = AKB \sin \omega t + AB_PK \qquad 8.73$$

where K = constant representing the dynamical Joule magnetostriction effect,

A = cross-sectional area of the rod, in square centimeters,

B_P = polarizing flux, in gausses,

B = amplitude of the alternating flux, in gausses,

$\omega = 2\pi f$,

f = frequency, in cycles per second, and

t = time, in seconds.

The relation between flux and the current is

$$B = \frac{4\pi Ni}{RA} \tag{8.74}$$

where N = number of turns,
i = current in the coil, in abamperes,
R = reluctance of the magnetic circuit, and
A = area of the rod, in square centimeters.

Combining equations 8.73 and 8.74 and eliminating the steady force,

$$f_M = \frac{4\pi NiK}{R} \sin \omega t \tag{8.75}$$

where K = constant representing the dynamical Joule magnetostriction effect,
R = reluctance of the magnetic circuit,
N = number of turns in the coil,
i = current, in abamperes,
$\omega = 2\pi f$,
f = frequency, in cycles per second, and
t = time, in seconds.

If the rod is allowed to vibrate, this stress may be considered to be the driving force.

The electromotive force, in abvolts, induced in the winding, due to the Villari effect, is

$$e = NA\frac{dB}{dt} \tag{8.76}$$

where N = number of turns,
A = cross-sectional area of the nickel rod, in square centimeters, and
B = magnetic induction, in gausses.

The magnetic induction is

$$B = \frac{4\pi K}{AR} x \tag{8.77}$$

where x = total extension of deformation, in centimeters, and
K = constant representing the dynamical Villari magnetostriction effect.

The induced voltage, in abvolts, is

$$e = \frac{4\pi NK}{R} \dot{x} \tag{8.78}$$

From equation 8.75

$$i = \frac{f_M R}{4\pi NK}$$

$$\frac{e}{i} = \frac{16\pi^2 N^2 K^2}{R^2} \frac{\dot{x}}{f_M} \tag{8.79}$$

In the above consideration it has been assumed that the stress and driving force are uniform over the length of the rod. Under these conditions the rod is a compliance given by

$$C_{M1} = \frac{l}{EA} \tag{8.80}$$

where A = cross-sectional area of the rod, in square centimeters,
l = length of the rod, in centimeters, and
E = Young's modulus.

The mechanical rectilineal impedance of the rod is

$$z_{M1} = \frac{1}{j\omega C_{M1}} \tag{8.81}$$

For the conditions under consideration the mechanical rectilineal impedance of the vibrating system, from the mechanical circuit of Fig. 8.6, is

$$z_M = z_{M1} + z_{M2} \tag{8.82}$$

where z_M = total mechanical rectilineal impedance, in mechanical ohms,
z_{M1} = mechanical rectilineal impedance of the rod, in mechanical ohms, and
z_{M2} = mechanical rectilineal impedance of the load, in mechanical ohms.

The mechanical rectilineal impedances z_M, z_{M1} and z_{M2} are referred to one end of the rod with the other end rigidly fixed. The dimensions of the rod are assumed to be small compared to the wavelength.

The mechanical rectilineal impedance at the end of the rod is

$$z_M = \frac{f_M}{\dot{x}} \qquad 8.83$$

From equations 8.79, 8.82 and 8.83

$$z_{EM} = \frac{16\pi^2 N^2 K^2}{R^2 z_M} \qquad 8.84$$

where z_{EM} = motional electrical impedance, in abohms, and
z_M = total mechanical rectilineal impedance load upon the rod, including the effective mechanical rectilineal impedance of the rod, in mechanical ohms.

The normal impedance of the coil is

$$z_{EN} = z_{E1} + z_{EM} \qquad 8.85$$

where z_{E1} = damped impedance of the voice coil, in abohms, and
z_{EM} = motional impedance, in abohms—equation 8.84.

The damped impedance of the coil of most magnetostriction systems comprises a resistance in series with an inductance (Fig. 8.6). The damped impedance and the motional impedance are effectively in series, as shown by equation 8.85 and depicted by the electrical circuit in Fig. 8.6.

In the above considerations the length of the rod is assumed to be a small fraction of the wavelength. In general, magnetostriction driving systems [2] are operated at resonance. The three most common systems are as follows: a rod fixed on one end and loaded on the other, a rod free on one end and loaded on the other and a free rod. The lumped constant representations of the three systems depicted by the mechanical networks in Fig. 8.7 are valid near the resonant frequency of the rod.

The mass m_1 in Fig. 8.7 is given by

$$m_1 = \frac{\rho l A}{2} \qquad 8.86$$

[2] Mason, "Electromechanical Transducers and Wave Filters," D. Van Nostrand Co., Princeton, N. J., 1946.

due to resistance, that is, air load and support resistance. This load is designated as the mechanical rectilineal resistance r_M in Fig. 8.7*C*.

The vibrating systems *A* and *B* given in Fig. 8.7 are usually employed to produce sound waves in liquids or gases. The vibrating system of Fig. 8.7*C* is usually employed as an element in a filter or as a frequency standard. For the latter use it is important that the load be very small.

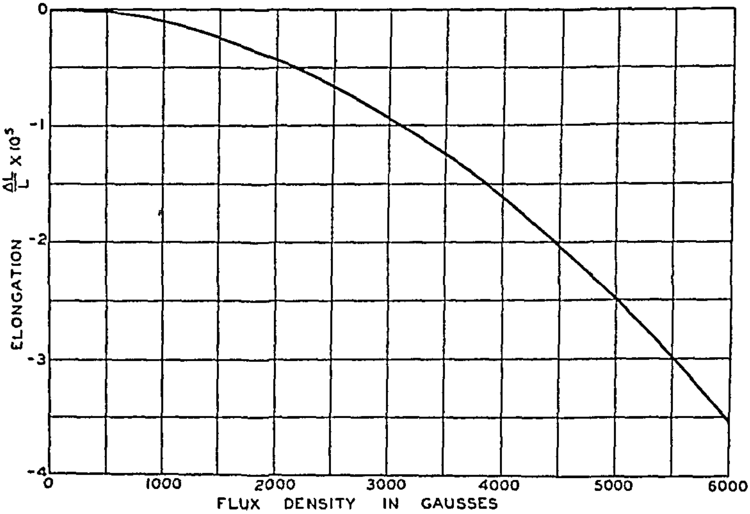

FIG. 8.8. Magnetostriction in nickel.

The mechanical rectilineal impedance z_M at f_M can be obtained from the mechanical networks of Fig. 8.7. The motional electrical impedance z_{EM} can be obtained from equation 8.84. The normal electrical impedance can then be determined from the electrical circuits of Fig. 8.7.

The magnetostrictive constant may be determined from the deformation-flux density characteristic. The elongation per unit length as a function of the flux density for nickel is shown in Fig. 8.8.

The deformation per unit length, due to a force, is

$$x = \frac{f_M}{EA} \tag{8.88}$$

where f_M = total force, in dynes,
A = area, in square centimeters, and
E = Young's modulus.

The magnetostrictive force is

$$f_M = KAB \tag{8.89}$$

where K = magnetostriction constant,
B = flux density, and
A = area, in square centimeters.

From equations 8.88 and 8.89 the deformation per unit length is

$$x = \frac{KB}{E} \tag{8.90}$$

The magnetostrictive constant K can be determined from the above equation, the data of Fig. 8.8 and Young's modulus.

8.6. Piezoelectric Driving System

A piezoelectric driving system is a driving system in which the mechanical forces result from the deformation of a crystal having converse

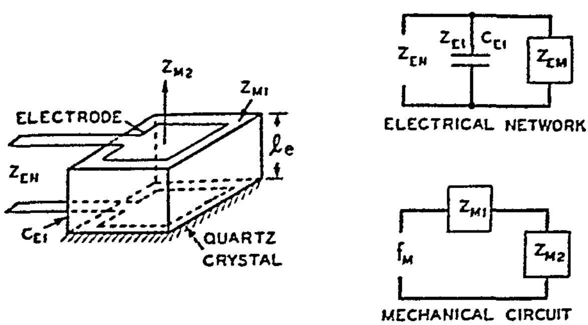

Fig. 8.9. Piezoelectric driving system. In the electrical network: z_{EN}, the normal electrical impedance of the crystal. z_{EM}, the motional electrical impedance of the crystal. z_{E1}, the damped electrical impedance of the crystal. $z_{E1} = 1/j\omega C_{E1}$. C_{E1}, the damped electrical capacitance of the crystal. In the mechanical circuit: f_M, the driving force. z_{M1}, the mechanical rectilineal impedance of the crystal. z_{M2}, the mechanical rectilineal impedance of the load.

piezoelectric properties. Among the crystals that exhibit piezoelectric phenomena are quartz, tourmaline, and Rochelle salt.

The consideration will be that of an X cut quartz crystal (Fig. 8.9).

The charge, in statcoulombs, due to the application of a force, is

$$q = Kf_M \tag{8.91}$$

where K = constant of the crystal, 6.4×10^{-8} for quartz, and
f_M = force, in dynes.

The displacement, in centimeters, due to an applied force, is

$$x = \frac{f_M l_e}{EA} \tag{8.92}$$

where f_M = force, in dynes,
l_e = length of the crystal, in centimeters,
E = Young's modulus, and
A = cross-sectional area, in square centimeters.

From equations 8.91 and 8.92

$$x = \frac{l_e q}{KEA} \tag{8.93}$$

Differentiating equation 8.93,

$$\dot{x} = \frac{l_e i}{KEA} \tag{8.94}$$

The deflection, in centimeters, due to the application of a voltage, is

$$x = Ke \tag{8.95}$$

where K = constant of the crystal, 6.4×10^{-8} for quartz, and
e = applied voltage, in statvolts.

From equations 8.92 and 8.95

$$f_M = \frac{KEAe}{l_e} \tag{8.96}$$

From equations 8.94 and 8.96

$$\frac{e}{i} = \frac{l_e^2 f_M}{K^2 E^2 A^2 \dot{x}} \tag{8.97}$$

In the above consideration it has been assumed that the stress and driving force are uniform over the length l_e of the crystal. Under these conditions the crystal is a compliance given by

$$C_{M1} = \frac{l_e}{EA} \qquad 8.98$$

where A = cross-sectional area of the crystal, in square centimeters,
l = length of the crystal, in centimeters, and
E = Young's modulus.

The mechanical rectilineal impedance of the crystal is

$$z_{M1} = \frac{1}{j\omega C_{M1}} \qquad 8.99$$

For the conditions under consideration the mechanical rectilineal impedance of the vibrating system, from the mechanical circuit of Fig. 8.9, is

$$z_M = z_{M1} + z_{M2} \qquad 8.100$$

where z_M = total mechanical rectilineal impedance, in mechanical ohms,
z_{M1} = mechanical rectilineal impedance of the crystal, in mechanical ohms, and
z_{M2} = mechanical rectilineal impedance of the load, in mechanical ohms.

The mechanical rectilineal impedances z_M, z_{M1} and z_{M2} are referred to one end of the crystal with the other end rigidly fixed. The dimensions of the crystal are assumed to be small compared to the wavelength.

The mechanical rectilineal impedance at the end of the crystal is

$$z_M = \frac{f_M}{\dot{x}} \qquad 8.101$$

From equations 8.97, 8.100 and 8.101

$$z_{EM} = \frac{l_e^2}{K^2E^2A^2} z_M \qquad 8.102$$

where z_{EM} = motional electrical impedance, in statohms, and
z_M = total mechanical rectilineal impedance including the crystal.

The normal electrical impedance of the crystal system is

$$z_{EN} = \frac{z_{EM}}{1 + j\omega C_{E1} z_{EM}} \tag{8.103}$$

where z_{EM} = motional impedance, equation 8.102, and
C_{E1} = capacitance of the crystal in the absence of motion.

The damped impedance and the motional impedance are effectively in parallel as shown by equation 8.103 and depicted by the electrical circuit in Fig. 8.9.

In the above considerations the length of the crystal is assumed to be a small fraction of the wavelength. In general, piezoelectric driving systems are operated at resonance. The three most common systems [3] are as follows: a crystal fixed on one end and loaded on the other, a crystal free on one end and loaded on the other and a free crystal. The lumped constant representations of the three systems depicted by the mechanical networks in Fig. 8.10 are valid near the resonant frequency of the crystal.

The mass m_1, in Fig. 8.10, is given by

$$m_1 = \frac{\rho l_e A}{2} \tag{8.104}$$

where ρ = density of the crystal, in grams per cubic centimeter,
l_e = length of the crystal, in centimeters, and
A = cross-sectional area of the crystal, in square centimeters.

The compliance C_{M1}, in Fig. 8.10, is given by

$$C_{M1} = \frac{8 l_e}{\pi^2 E A} \tag{8.105}$$

where A = cross-sectional area of the crystal, in square centimeters,
l_e = length of the crystal, in centimeters, and
E = Young's modulus.

The compliance given by equation 8.105 is $8/\pi^2$ times the static compliance given by equation 8.98.

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[3] Mason, "Electromechanical Transducers and Wave Filters," D. Van Nostrand Co., Princeton, N. J., 1946.

The load on the end of the crystal is the mechanical rectilineal impedance z_{M2}. In the case of a free crystal, Fig. 8.10C, the only load is the dissipation due to resistance, that is, air load and support resistance. This load is designated as the mechanical rectilineal resistance r_M in Fig. 8.10C.

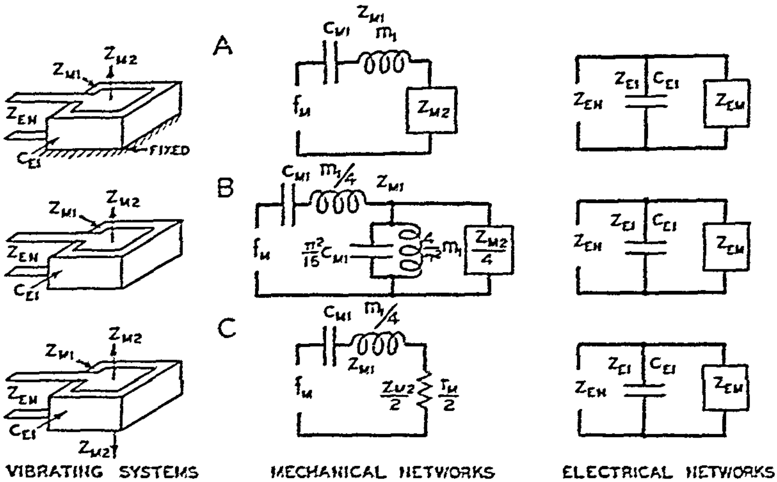

FIG. 8.10. Piezoelectric driving systems. A. Crystal fixed on one end and loaded on the other. B. Crystal free on one end and loaded on the other. C. Free crystal, that is, a light load on both ends. In the electrical networks: z_{EN}, the normal electrical impedance of the crystal. z_{EM}, the motional electrical impedance of the crystal. z_{E1}, the damped electrical impedance of the crystal. $z_{E1} = 1/j\omega C_{E1}$. C_{E1}, the damped electrical capacitance of the crystal. In the mechanical networks: f_M, the driving force. z_{M2}, the mechanical rectilineal impedance of the mechanical load. z_{M1}, the mechanical rectilineal impedance of the crystal. m_1 and C_{M1}, the effective mass and compliance of the crystal.

The vibrating systems *A* and *B* in Fig. 8.10 are usually employed to produce sound waves in liquids or gases. The vibrating system of Fig. 8.10C is usually employed as an element in a filter or as a frequency standard. For the latter use it is important that the load be very small.

The mechanical rectilineal impedance z_M at f_M can be obtained from the mechanical networks of Fig. 8.10. The motional electrical impedance z_{EM} can be obtained from equation 8.102. The normal electrical impedance can then be determined from the electrical networks of Fig. 8.10.

CHAPTER IX

GENERATING SYSTEMS

9.1. Introduction

A mechanical electrical generating system is a system for converting mechanical or acoustical vibrations into the corresponding electrical variations. The most common generating systems in use to-day for converting mechanical vibrations into the corresponding electrical variations are the electrodynamic, the electromagnetic, the electrostatic, the piezoelectric and the magnetostriction. It is the purpose of this chapter to describe the electrical and mechanical characteristics of these generating systems.

9.2. Electrodynamic Generating System

A moving conductor or a moving coil generating system is a generating system in which the electromotive force is developed by motion of a conductor through a magnetic field.

The voltage, in abvolts, due to the motion of the conductor in the magnetic field, Fig. 9.1, is

$$e = Bl\dot{x} \qquad 9.1$$

where B = flux density, in gausses,
l = length of the conductor, in centimeters, and
$\dot{x}$ = velocity of the conductor, in centimeters per second.

The velocity of the conductor is governed by the mechanical driving force, the mechanical rectilineal impedance of the mechanical system, and the mechanical rectilineal impedance due to the electrical system. The vibrating system is shown in Fig. 9.1. In the mechanical circuit [1] z_M represents the mechanical rectilineal impedance of the mechanical portion of the vibrating system actuated by f_M including the mechanical rectilineal impedance of the coil at the voice coil. f_M represents the

[1] See footnote 1, page 131.

mechanomotive force at the voice coil. The mechanical rectilineal impedance due to the electrical system from equation 8.7 of the chapter on Driving Systems, is

$$z_{ME} = \frac{(Bl)^2}{z_E} \qquad 9.2$$

where B = flux density, in gausses,
l = length of the conductor, in centimeters,
$z_E = z_{E1} + z_{E2}$,
z_{E1} = electrical impedance of the voice coil, in abohms, and
z_{E2} = electrical impedance of the external load, in abohms.

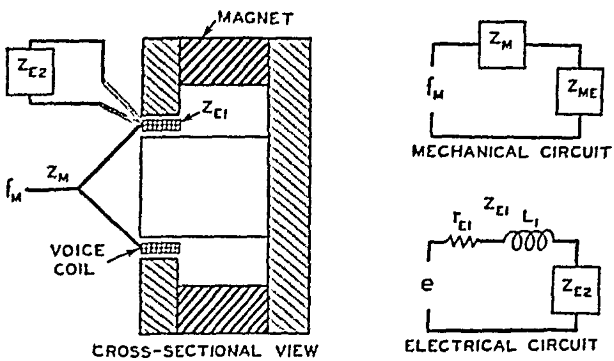

Fig. 9.1. Electrodynamic generating system. In the mechanical circuit: f_M, the external driving force. z_M, the total mechanical rectilineal impedance of the mechanical portion of the vibrating system actuated by f_M. z_{ME}, the mechanical rectilineal impedance due to the electrical circuit. In the electrical circuit: e, the internal electromotive force generated in the voice coil. z_{E1}, the damped electrical impedance of the voice coil. $z_{E1} = r_{E1} + j\omega L_1$. L_1 and r_{E1}, the damped inductance and electrical resistance of the voice coil. z_{E2}, the electrical impedance of the external load.

The velocity of the voice coil is

$$\dot{x} = \frac{f_M}{z_M + z_{ME}} \qquad 9.3$$

From equations 9.1 and 9.3 the generated electromotive force, in abvolts, is

$$e = Bl\dot{x} = \frac{Blf_M}{z_M + z_{ME}} \qquad 9.4$$

The generated electromotive force is effectively in series with the electrical impedance z_{E1} of the voice coil and the electrical impedance z_{E2} of the external load, as depicted by the electrical circuit in Fig. 9.1.

9.3. Electromagnetic Generating Systems

A magnetic generating system is a generating system in which the electromotive force is developed by the charge in magnetic flux through a stationary coil. There are two general types of magnetic generating systems; namely, the reed armature type and the balanced armature type.

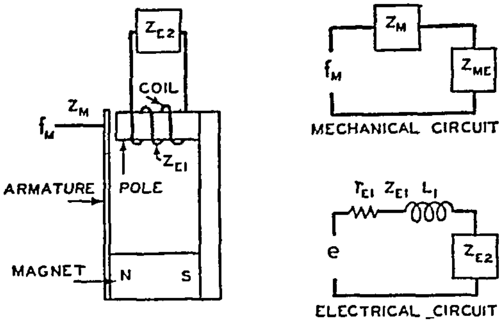

FIG. 9.2. Reed armature electromagnetic generating system. In the mechanical circuit: f_M, the external driving force. z_M, the total mechanical rectilineal impedance of the mechanical portion of the vibrating system actuated by f_M. z_{ME}, the mechanical rectilineal impedance due to the electrical circuit: In the electrical circuit: e, the internal electromotive force generated in the coil. z_{E1}, the damped electrical impedance of the coil. $z_{E1} = r_{E1} + j\omega L_1$. L_1 and r_{E1}, the damped inductance and electrical resistance of the coil. z_{E2}, the electrical impedance of the external load.

A. *Reed Armature Generating System.*—The reed armature generating system, shown in Fig. 9.2, consists of a coil around a polarized magnetic field in which the reluctance is varied by an armature of steel. The variation of reluctance causes a corresponding change in magnetic flux through the coil and thereby leads to the induction of an electromotive force.

The electromotive force, in abvolts, generated in the coil due to motion of the armature from equation 8.32 of the chapter on Driving Systems, is

$$e = \frac{NMA}{a^2} \dot{x} \qquad 9.5$$

where N = number of turns in the coil,
M = magnetomotive force, in gilberts, due to the steady field,
A = area of the pole, in square centimeters,
a = spacing between the armature and pole, in centimeters, and
$\dot{x}$ = velocity of the armature, in centimeters per second.

The velocity of the armature is governed by the mechanical driving force, the mechanical rectilineal impedance of the mechanical system, and the mechanical rectilineal impedance due to the electrical system. The vibrating system is shown in Fig. 9.2. In the mechanical circuit z_M represents the mechanical rectilineal impedance of the mechanical portion of the vibrating system actuated by f_M including the mechanical rectilineal impedance of the armature. f_M represents the mechanomotive force at the armature. The mechanical rectilineal impedance due to the electrical system from equation 8.38 of the chapter on Driving Systems is

$$z_{ME} = \frac{\phi_1{}^2 N^2}{a^2 z_E} \qquad 9.6$$

where ϕ_1 = total flux, in maxwells, through the armature,
N = number of turns on the coil,
a = spacing between the armature and pole, in centimeters, and
$z_E = z_{E1} + z_{E2}$,
z_{E1} = electrical impedance of the coil, in abohms, and
z_{E2} = electrical impedance of the external load, in abohms.

The velocity of the armature, in centimeters per second, is

$$\dot{x} = \frac{f_M}{z_M + z_{ME}} \qquad 9.7$$

From equations 9.5 and 9.7

$$e = \frac{NMA}{a^2}\dot{x} = \frac{NMAf_M}{a^2(z_M + z_{ME})} \qquad 9.8$$

The generated electromotive force is effectively in series with the electrical impedance z_{E1} of the coil and the electrical impedance z_{E2} of the external load as depicted by the electrical circuit in Fig. 9.2.

B. *Balanced Armature Generating System.*—In the simple generating system of the preceding section both the steady magnetic flux and the change in flux, due to the deflection of the armature, flows through the armature. Consider a balanced armature type of generating system in which only the alternating flux flows longitudinally through the armature as shown in Fig. 9.3.

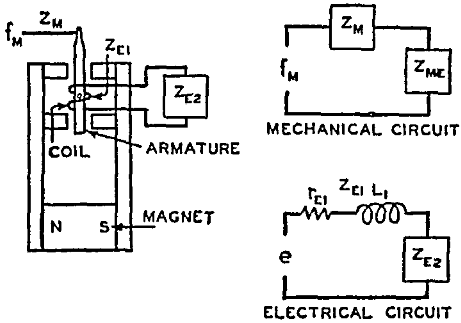

FIG. 9.3. Balanced armature electromagnetic generating system. In the mechanical circuit: z_M, the total mechanical rectilineal impedance of the mechanical portion of the vibrating system actuated by f_M. z_{ME}, the mechanical rectilineal impedance due to the electrical circuit. In the electrical circuit: e, the internal electromotive force generated in the coil. z_{E1}, the damped electrical impedance of the coil. $z_{E1} = r_{E1} + j\omega L_1$. L_1 and r_{E1}, the damped inductance and electrical resistance of the coil. z_{E2}, the electrical impedance of the external load.

The electromotive force, in abvolts, generated in the coil due to motion of the armature from equation 8.48 of the chapter on Driving Systems is

$$e = \frac{NMA}{a^2}\dot{x} \tag{9.9}$$

where N = number of turns in the coil,
M = magnetomotive force, in gilberts, of the steady field,
A = area of a pole piece, in square centimeters,
a = spacing between the armature and pole, in centimeters, and
$\dot{x}$ = velocity of the armature, in centimeters per second.

The velocity of the armature is governed by the mechanical driving force, the mechanical rectilineal impedance of the mechanical system, and the mechanical rectilineal impedance due to the electrical system.

The vibrating system is shown in Fig. 9.3. In the mechanical circuit z_M represents the mechanical rectilineal impedance of the mechanical portion of the vibrating system including the mechanical rectilineal impedance of the armature. f_M represents the mechanomotive force on the armature. The mechanical rectilineal impedance due to the electrical system from equation 8.52 of the chapter on Driving Systems is

$$z_{ME} = \frac{4N^2\phi M}{a^2 R_2 z_E} \tag{9.10}$$

where N = number of turns in the coil,
ϕ = total flux in the air gap at one of the poles, in maxwells,
M = magnetomotive force, in gilberts, of the magnet,
a = spacing between armature and pole, in centimeters,
R_2 = reluctance, in oersteds, of the alternating magnetic circuit,
$z_E = z_{E1} + z_{E2}$,
z_{E1} = electrical impedance of the coil, in abohms, and
z_{E2} = electrical impedance of the external load, in abohms.

The velocity of the armature, in centimeters per second, is

$$\dot{x} = \frac{f_M}{z_M + z_{ME}} \tag{9.11}$$

From equations 9.9 and 9.11

$$e = \frac{NMA}{a^2}\dot{x} = \frac{NMAf_M}{a^2(z_M + z_{ME})} \tag{9.12}$$

The generated electromotive force is effectively in series with the electrical impedance z_{E1} of the coil and the electrical impedance z_{E2} of the external load, as depicted by the electrical circuit in Fig. 9.3.

9.4. Electrostatic Generating System

A condenser or electrostatic generating system is a generating system in which the electromotive force is developed by the relative motion between two differently electrostatically charged plates.

The current, in statamperes, generated by the motion of the movable plate of the condenser from equation 8.67 of the chapter on Driving Systems is

$$i_1 = \frac{e_0 A\dot{x}}{4\pi a^2} \tag{9.13}$$

where e = polarizing voltage, in statvolts,
A = area of the plate, in square centimeters,
a = spacing between the plates, in centimeters, and
$\dot{x}$ = velocity of the movable plate, in centimeters per second.

The current, in statamperes, due to the electromotive force e across the electrical impedances z_{E1} and z_{E2} of Fig. 9.4 is

$$i_2 = -\frac{e}{z_E} \tag{9.14}$$

where e = electromotive force, in statvolts,

$$z_E = \frac{z_{E1} z_{E2}}{z_{E1} + z_{E2}}$$

$$z_{E1} = \frac{1}{j\omega C_{E1}}$$

C_{E1} = electrical capacitance of the condenser, in statfarads, and
z_{E2} = electrical impedance of the external load, in statohms.

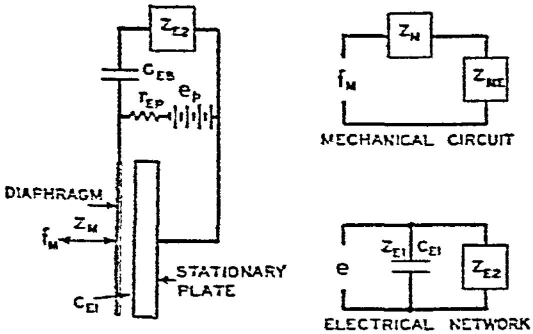

FIG. 9.4. Electrostatic generating system. In the mechanical circuit: z_M, the total mechanical rectilineal impedance of the mechanical portion of the vibrating system actuated by f_M. z_{ME}, the mechanical rectilineal impedance due to the electrical circuit. In the electrical network: e, the electromotive force generated across the condenser. z_{E1}, the damped electrical impedance of the condenser. $z_{E1} = 1/j\omega C_{E1}$. C_{E1}, damped electrical capacitance of the condenser. z_{E2}, the electrical impedance of the external load.

Since there is no external current applied on the electrical side of the system the sum of the currents i_1 and i_2 is zero. From equations 9.13

and 9.14 the generated electromotive force, e, in statvolts, across the electrical impedances z_{E1} and z_{E2} is

$$e = \frac{e_0 A \dot{x}}{4\pi a^2} z_E \qquad 9.15$$

The velocity of the movable plate is governed by the mechanical driving force, the mechanical rectilineal impedance of the mechanical system and the mechanical rectilineal impedance due to the electrical system. The vibrating system is shown in Fig. 9.4. In the mechanical circuit z_M represents the mechanical rectilineal impedance of the mechanical portion of the vibrating system actuated by f_M including the mechanical impedance of the movable plate. f_M represents the mechanomotive force at the movable plate. The mechanical rectilineal impedance due to the electrical system from equation 8.71 of the chapter on Driving Systems is

$$z_{ME} = \frac{e_0^2 A^2}{16\pi^2 a^4} z_E \qquad 9.16$$

where e_0 = polarizing voltage, in statvolts,
a = spacing between plates, in centimeters,
A = area of the plates, in square centimeters,

$$z_E = \frac{z_{E1} z_{E2}}{z_{E1} + z_{E2}}$$

$$z_{E1} = \frac{1}{j\omega C_{E1}}$$

C_{E1} = capacity of the generator, in statfarads,
z_{E2} = electrical impedance of the external load, in statohms.

The velocity of the movable plate, in centimeters per second, is

$$\dot{x} = \frac{f_M}{z_M + z_{ME}} \qquad 9.17$$

From equations 9.15 and 9.17 the electromotive force e across z_{E1} and z_{E2} in parallel, depicted by the electrical network of Fig. 9.4, is

$$e = \frac{e A z_E f_M}{4\pi a^2 (z_M + z_{ME})} \qquad 9.18$$

The electromotive force e_1 in series with z_{E1} and z_{E2} which will produce the electromotive force e across z_{E2} is of interest in the design of generating systems.

Equation 9.15 may be written

$$e = \frac{e_0 A\dot{x}}{4\pi a^2}\left(\frac{z_{E1}z_{E2}}{z_{E1} + z_{E2}}\right) \qquad 9.19$$

The electrical capacitance of the condenser C_{E1} from equation 8.62 of the chapter on Driving Systems is

$$C_{E1} = \frac{A}{4\pi a} \qquad 9.20$$

The electrical impedance z_{E1} is

$$z_{E1} = \frac{1}{j\omega C_{E1}} = \frac{4\pi a}{j\omega A} \qquad 9.21$$

Substituting equation 9.21 in 9.19,

$$e = \frac{e_0 \dot{x}}{aj\omega}\left(\frac{z_{E2}}{z_{E1} + z_{E2}}\right) \qquad 9.22$$

The amplitude in terms of the velocity is

$$x = \frac{\dot{x}}{j\omega} \qquad 9.23$$

Substituting equation 9.23 in 9.22,

$$e = \frac{e_0 x}{a}\left(\frac{z_{E2}}{z_{E1} + z_{E2}}\right) \qquad 9.24$$

The electromotive force e in terms of e_1 and the impedances z_{E1} and z_{E2} is

$$e = \frac{e_1 z_{E2}}{z_{E1} + z_{E2}} \qquad 9.25$$

Comparing equations 9.24 and 9.25,

$$e_1 = \frac{e_0 x}{a} \qquad 9.26$$

The electrostatic generating system may be considered to consist of a generator having an internal or open circuit electromotive force e_1 as given by equation 9.26 and an internal impedance z_{E1}. Equation 9.26 shows that this electromotive force is independent of the frequency if the amplitude is independent of the frequency. However, the voltage e across the load may vary with frequency depending upon the nature of load z_{E2}.

9.5. Magnetostriction Generating System

A magnetostriction generating system is a generating system in which the electromotive force is developed in a stationary coil by a change in

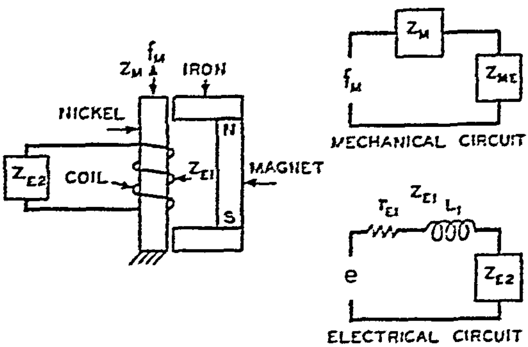

FIG. 9.5. Magnetostriction generating system. In the mechanical circuit: z_M, the total mechanical rectilineal impedance of the mechanical portion of the vibrating system actuated by f_M. z_{ME}, the mechanical rectilineal impedance due to the electrical circuit. In the electrical circuit: e, the internal electromotive force generated in the coil. z_{E1}, the damped electrical impedance of the coil. $z_{E1} = r_{E1} + j\omega L_1$. L_1 and r_{E1}, the damped inductance and electrical resistance of the coil. z_{E2}, the electrical impedance of the external load.

magnetic flux due to the deformation of a ferromagnetic material having magnetostriction properties. The magnetostriction generator, shown in Fig. 9.5, consists of a coil surrounding a magnetic circuit which includes a ferromagnetic material having magnetostriction properties. The voltage, in abvolts, developed in the coil due to deformation of the rod, from equation 8.78 of the chapter on Driving Systems, is

$$e = \frac{4\pi NK}{R} \dot{x} \qquad 9.27$$

where N = number of turns in the coil,
R = reluctance of the magnetic circuit,
K = constant representing the dynamical Villari magnetostriction effect, and
$\dot{x}$ = velocity at the point of application of the driving force to the rod, in centimeters per second.

The velocity of the rod is governed by the mechanical driving force, the mechanical impedance of the mechanical system and the mechanical impedance due to the electrical system. The vibrating system is shown in Fig. 9.5. In the mechanical circuit z_M represents the mechanical rectilineal impedance of the mechanical portion of the vibrating system actuated by f_M including the mechanical rectilineal impedance of the magnetostriction rod. f_M represents the mechanomotive force on the rod. It is assumed that the force f_M is the same at all points along the length of the rod and that the phase of the amplitude is constant along the rod. The mechanical rectilineal impedance due to the electrical system from equation 8.79 or 8.84 of the chapter on Driving Systems is

$$z_{ME} = \frac{16\pi^2 N^2 K^2}{z_E R^2} \qquad 9.28$$

where N = number of turns in the coil,
K = magnetostriction constant,
R = reluctance of the magnetic circuit, and
$z_E = z_{E1} + z_{E2}$,
z_{E1} = electrical impedance of the coil, in abohms, and
z_{E2} = electrical impedance of the external circuit, in abohms.

The dimensions of the rod are assumed to be a small fraction of a wavelength. Under these conditions the rod is a compliance given by

$$C_{M1} = \frac{l}{EA} \qquad 9.29$$

where A = cross-sectional area of the rod, in square centimeters,
l = length of the rod, in centimeters, and
E = Young's modulus.

The mechanical rectilineal impedance of the rod is

$$z_{M1} = \frac{1}{j\omega C_{M1}} \qquad 9.30$$

For the conditions under consideration the mechanical rectilineal impedance of the vibrating system is

$$z_M = z_{M1} + z_{M2} \qquad 9.31$$

where z_M = total mechanical rectilineal impedance, in mechanical ohms,
z_{M1} = mechanical rectilineal impedance of the rod, in mechanical ohms, and
z_{M2} = mechanical rectilineal impedance of the load, in mechanical ohms.

The velocity of the rod, in centimeters per second, at the driving point is

$$\dot{x} = \frac{f_M}{z_M + z_{ME}} \qquad 9.32$$

From equations 9.27 and 9.32 the generated electromotive force, in abvolts, is

$$e = \frac{4\pi NKf_M}{(z_M + z_{ME})R} \qquad 9.33$$

The generated electromotive force is effectively in series with the electrical impedance z_{E1} of the coil and the electrical impedance z_{E2} of the external load as depicted by the electrical circuit in Fig. 9.5.

In the above considerations the length of the rod is assumed to be a small fraction of the wavelength. In general, magnetostriction generating systems are operated at resonance. The two most common systems are as follows: a rod fixed on one end and driven on the other and a rod free on one end and driven on the other. The lumped constant representations of the two systems shown in Fig. 9.6, are valid near the resonant frequency of the rod. The mass m_1 and compliance C_{M1}, in Fig. 9.6, are given by equations 8.86 and 8.87 in the chapter on Driving Systems. The load on the end of the rod is the mechanical rectilineal impedance z_{M2}. The mechanical rectilineal impedance z_{ME} due to the electrical circuit is given by equation 9.28. From the above constants and the driving force f_M the velocity in the mechanical circuit can be

determined. The open circuit electromotive force e of the electrical circuit of Fig. 9.6 can be obtained from equation 9.27 and the velocity.

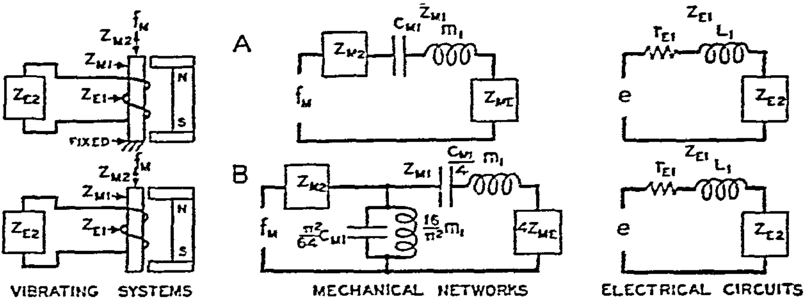

FIG. 9.6. Magnetostriction generating systems. A. Rod fixed on one end and driven on the other. B. Rod free on one end and driven on the other. In the mechanical networks: f_M, the driving force. z_{ME}, the mechanical rectilineal impedance due to the electrical circuit. z_{M2}, the mechanical rectilineal impedance of the external mechanical load. z_{M1}, the mechanical rectilineal impedance of the rod. m_1 and C_{M1}, the effective mass and compliance of the rod. In the electrical circuits: e, the internal electromotive force generated in the coil. z_{E1}, the damped electrical impedance of the coil. $z_{E1} = r_{E1} + j\omega L_1$. L_1 and r_{E1}, the damped inductance and electrical resistance of the coil, z_{E2}, the electrical impedance of the external electrical load.

9.6. Piezoelectric Generating System

A piezoelectric generating system is a generating system in which the electromotive force is developed by the deformation of a crystal having converse piezoelectric properties. The crystal generating system, shown in Fig. 9.7, consists of a suitably ground crystal having converse piezoelectric properties fitted with appropriate electrodes.

The current, in statamperes, generated by the motion of the crystal from equation 8.94 of the chapter on Driving Systems is

$$i_1 = \frac{KEA}{l_c}\dot{x} \tag{9.34}$$

where K = constant of the crystal, 6.4×10^{-8} for quartz,
E = Young's modulus,
l_c = length of the crystal, in centimeters,
A = cross-sectional area of the crystal, in square centimeters, length of the crystal, in centimeters, and
$\dot{x}$ = velocity of the crystal, in centimeters per second.

The current, in statamperes, due to the electromotive force e across the electrical impedances z_{E1} and z_{E2} is

$$i_2 = -\frac{e}{z_E} \tag{9.35}$$

where e = electromotive force, in statvolts,

$z_E = \dfrac{z_{E1}z_{E2}}{z_{E1} + z_{E2}}$

$z_{E1} = \dfrac{1}{j\omega C_{E1}}$

C_{E1} = electrical capacitance of the crystal, in statfarads, and

z_{E2} = electrical impedance of the external load, in statohms.

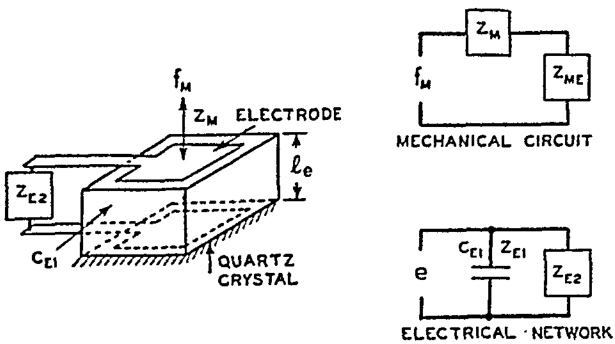

Fig. 9.7. Piezoelectric generating system. In the electrical circuit: z_M, the total mechanical rectilineal impedance of the mechanical portion of the vibrating system actuated by f_M. z_{ME}, the mechanical rectilineal impedance due to the electrical network. In the electrical circuit: e, the electromotive force generated across the crystal. z_{E1}, the damped electrical impedance of the crystal. $z_{E1} = 1/j\omega C_{E1}$. C_{E1}, the damped electrical capacitance of the crystal. z_{E2}, the electrical impedance of the external load.

Since there is no external current applied to the electrodes of the crystal the sum of the currents i_1 and i_2 is zero. From equations 9.34 and 9.35 the generated electromotive force e, in statvolts, across the electrical impedance z_{E1} and z_{E2} is

$$e = \frac{KEA\dot{x}}{l_e} z_E \tag{9.36}$$

The velocity at the end of the crystal is governed by the mechanical driving force, the mechanical rectilineal impedance of the mechanical

system and the mechanical rectilineal impedance due to the electrical system. The vibrating system is shown in Fig. 9.7. In the mechanical circuit z_M represents the mechanical rectilineal impedance of the mechanical portion of the vibrating system actuated by f_M including the mechanical rectilineal impedance of the crystal. f_M represents the mechanomotive force at the end of the crystal. It is assumed that the force f_M is the same at all points along the length of the crystal and that the phase of the amplitude is constant along the crystal. The mechanical impedance due to the electrical system from equation 8.97 or 8.102 of the chapter on Driving Systems is

$$z_{ME} = \frac{K^2E^2A^2}{l_e^2} z_E \tag{9.37}$$

where K = constant of the crystal 6.4×10^{-8} for quartz,
E = Young's modulus,
l_e = length of the crystal, in centimeters,
A = area of the electrode, in centimeters, length of the crystal, in centimeters, and

$$z_E = \frac{z_{E1}z_{E2}}{z_{E1} + z_{E2}} \tag{9.38}$$

$$z_{E1} = \frac{1}{j\omega C_{E1}} \tag{9.39}$$

C_{E1} = capacitance of the generator, in statfarads,
z_{E2} = electrical impedance of the external load, in statohms.

The dimensions of the crystal are assumed to be a small fraction of a wavelength. Under these conditions the crystal is a compliance given by

$$C_{M1} = \frac{l_e}{EA} \tag{9.40}$$

where A = cross-sectional area of the crystal, in square centimeters,
l_e = length of the crystal, in centimeters, and
E = Young's modulus.

The mechanical rectilineal impedance of the crystal is

$$z_{M1} = \frac{1}{j\omega C_{M1}} \tag{9.41}$$

For the conditions under consideration the mechanical rectilineal impedance of the vibrating system is

$$z_M = z_{M1} + z_{M2} \qquad 9.42$$

where z_M = total mechanical rectilineal impedance, in mechanical ohms,
z_{M1} = mechanical rectilineal impedance of the crystal, in mechanical ohms, and
z_{M2} = mechanical rectilineal impedance of the load, in mechanical ohms.

The velocity at the end of crystal, in centimeters per second, is

$$\dot{x} = \frac{f_M}{z_M + z_{ME}} \qquad 9.43$$

From equations 9.36 and 9.43 the electromotive force across z_{E1} and z_{E2} in parallel, depicted by the electrical network of Fig. 9.7, is

$$e = \frac{KEAz_E f_M}{l_e(z_M + z_{ME})} \qquad 9.44$$

The electromotive force e_1 in series with z_{E1} and z_{E2} which will produce the electromotive force e across z_{E2} is of interest in the design of generating systems.

Equation 9.36 may be written

$$e = \frac{KEA\dot{x}}{l_e}\left(\frac{z_{E1}z_{E2}}{z_{E1} + z_{E2}}\right) \qquad 9.45$$

The electrical capacitance of the crystal is

$$C_{E1} = \frac{AD}{4\pi l_e} \qquad 9.46$$

where D = dielectric constant of the crystal.

The electrical impedance z_{E1} is

$$z_{E1} = \frac{1}{j\omega C_{E1}} = \frac{4\pi l_e}{j\omega AD} \qquad 9.47$$

Substituting 9.47 in 9.45,

$$e = \frac{4\pi KE\dot{x}}{j\omega D}\left(\frac{z_{E2}}{z_{E1} + z_{E2}}\right) \qquad 9.48$$

The amplitude in terms of the velocity is

$$x = \frac{\dot{x}}{j\omega} \qquad 9.49$$

Substituting 9.49 in 9.48,

$$e = \frac{4\pi KEx}{D}\left(\frac{z_{E2}}{z_{E1} + z_{E2}}\right) \qquad 9.50$$

The electromotive force e in terms of e_1 is

$$e = \frac{e_1 z_{E2}}{z_{E1} + z_{E2}} \qquad 9.51$$

Comparing equations 9.50 and 9.51,

$$e_1 = \frac{4\pi KEx}{D} \qquad 9.52$$

The piezoelectric generating system may be considered to consist of a generator having an internal or open circuit electromotive force e_1 as given by equation 9.52 and an internal impedance z_{E1}. Equation 9.52 shows that this electromotive force is independent of the frequency if the amplitude is independent of the frequency. However, the voltage e across the load may vary with frequency depending upon the nature of the load z_{E2}.

In the above considerations the length of the crystal is assumed to be a small fraction of the wavelength. In general piezoelectric generating systems are operated at resonance. The two most common systems are as follows: a crystal fixed on one end and driven on the other and a crystal free on one end and driven on the other. The lumped constant representations of the two systems shown in Fig. 9.8 are valid near the resonant frequency of the crystal. The mass m_1 and compliance C_{M1}, in Fig. 9.8, are given by equations 8.104 and 8.105 in the chapter on Driving Systems. The load on the end of the crystal is the mechanical rectilineal impedance z_{M2}. The mechanical rectilineal impedance z_{ME}

due to the electrical circuit is given by equation 9.37. From the above constants and the driving force f_M the velocity in the mechanical circuit can be determined. The electromotive e across the crystal of the

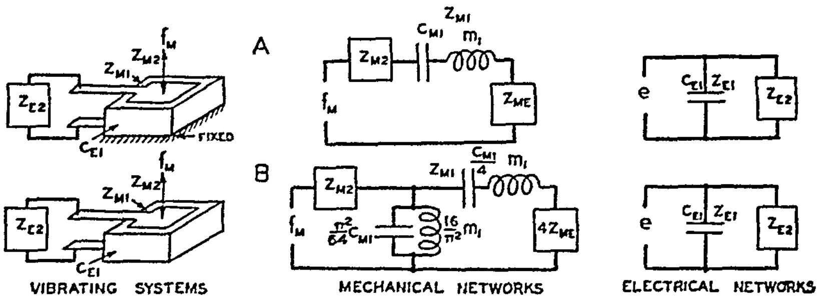

FIG. 9.8. Piezoelectric generating systems. A. Crystal fixed on one end and driven on the other. B. Crystal free on one end and driven on the other. In the mechanical networks: f_M, the driving force. z_{ME}, the mechanical rectilineal impedance due to the electrical circuit. z_{M2}, the mechanical rectilineal impedance of the external mechanical load. z_{M1}, the mechanical rectilineal impedance of the crystal. m_1 and C_{M1}, the effective mass and compliance of the crystal. In the electrical circuits: e, the electromotive force generated across the crystal. z_{E1}, the damped electrical impedance of the crystal. $z_{E1} = 1/j\omega C_{E1}$. C_{E1}, the damped electrical capacitance of the crystal. z_{E2}, the electrical impedance of the external load.

electrical network of Fig. 9.8 can be obtained from equation 9.45 and the velocity.

CHAPTER X

THEOREMS

10.1. Introduction

A number of dynamical laws common to electrical, mechanical rectilineal, mechanical rotational and acoustical systems have been described in this book. There are other dynamical laws that are well known in electrical circuit theory which can be applied to mechanical and acoustical systems. It is the purpose of this chapter to illustrate the application of reciprocity, Thevenin's and superposition theorems to electrical, mechanical rectilineal, mechanical rotational and acoustical systems.

10.2. Reciprocity Theorems [1]

A. *Electrical Reciprocity Theorem.*—In an electrical system composed of the electrical elements of inductance, electrical capacitance and electrical resistance, let a set of electromotive forces e_1', e_2', $e_3' \ldots e_n'$ all harmonic of the same frequency acting in n points in the invariable network, produce a current distribution i_1', i_2', $i_3' \ldots i_n'$, and let a second set of electromotive forces e_1'', e_2'', $e_3'' \ldots e_n''$ of the same frequency as the first produce a second current distribution i_1'', i_2'', $i_3'' \ldots i_n''$. Then

$$\sum_{j=1}^{n} e_j' i_j'' = \sum_{j=1}^{n} e_j'' i_j' \tag{10.1}$$

This theorem is valid provided the electrical system is invariable, contains no internal source of energy or unilateral device, linearity in the relations between electromotive forces and currents and complete reversibility in the elements, and provided the electromotive forces e_1, e_2, $e_3 \ldots e_n$ are all of the same frequency.

[1] Ballentine, S., *Proc. I.R.E.*, Vol. 17, No. 6, p. 929, 1929. "Reciprocity in Electromagnetic and Other Systems."

In the simple case in which there are only two electromotive forces, as illustrated in the electrical system of Fig. 10.1, equation 10.1 becomes

$$e'i'' = e''i' \qquad 10.2$$

where e', e'' and i', i'' are the electromotive forces and currents depicted in the electrical system of Fig. 10.1.

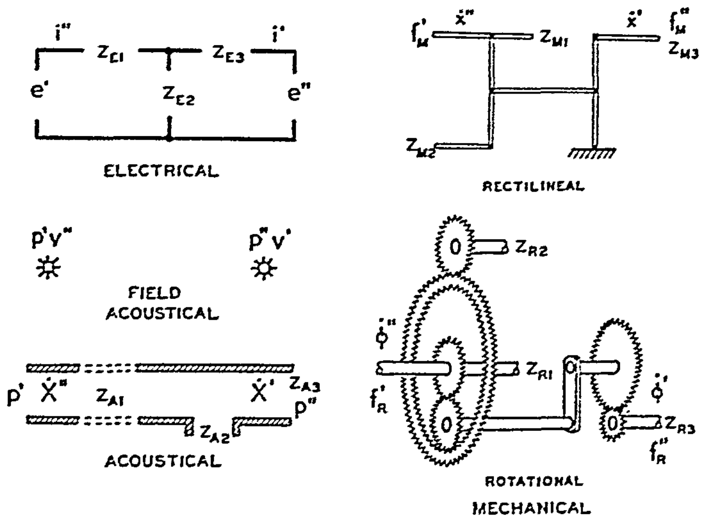

FIG. 10.1. Reciprocity in electrical, mechanical rectilineal, mechanical rotational and acoustical systems.

B. *Mechanical Rectilineal Reciprocity Theorem.*—In a mechanical rectilineal system composed of mechanical elements of mass, compliance and mechanical resistance, let a set of forces $f_{M1}', f_{M2}', f_{M3}' \ldots f_{Mn}'$ all harmonic of the same frequency acting in n points in the system produce a velocity distribution $\dot{x}_1', \dot{x}_2', \dot{x}_3' \ldots \dot{x}_n'$, and let a second set of forces $f_{M1}'', f_{M2}'', f_{M3}'' \ldots f_{Mn}''$ of the same frequency as the first produce a second velocity distribution $\dot{x}_1'', \dot{x}_2'', \dot{x}_3'' \ldots \dot{x}_n''$. Then

$$\sum_{j=1}^{n} f_{Mj}'\dot{x}_j'' = \sum_{j=1}^{n} f_{Mj}''\dot{x}_j' \qquad 10.3$$

This theorem is valid provided the mechanical system is invariable, contains no internal source of energy or unilateral device, linearity in the relations between forces and velocities and complete reversibility in

the elements, and provided the applied forces $f_{M1}, f_{M2}, f_{M3} \ldots f_{Mn}$ are all of the same frequency.

In the simple case in which there are only two forces, as illustrated in the mechanical rectilineal system of Fig. 10.1, equation 10.3 becomes

$$f_M'\dot{x}'' = f_M''\dot{x}' \qquad 10.4$$

where f_M', f_M'' and $\dot{x}', \dot{x}''$ are the forces and velocities depicted in the mechanical rectilineal system of Fig. 10.1.

C. *Mechanical Rotational Reciprocity Theorem.*—In a mechanical rotational system composed of mechanical rotational elements of moment of inertia, rotational compliance and mechanical rotational resistance, let a set of torques $f_{R1}', f_{R2}', f_{R3}' \ldots f_{Rn}'$ all harmonic of the same frequency acting in n points in the system, produce a rotational velocity distribution $\dot{\phi}_1', \dot{\phi}_2', \dot{\phi}_3' \ldots \dot{\phi}_n'$, and let a second set of torques f_{R1}'', $f_{R2}'', f_{R3}'' \ldots f_{Rn}''$ of the same frequency as the first produce a second rotational velocity distribution $\dot{\phi}_1'', \dot{\phi}_2'', \dot{\phi}_3'' \ldots \dot{\phi}_n''$. Then

$$\sum_{j=1}^{n} f_{Rj}'\dot{\phi}_j'' = \sum_{j=1}^{n} f_{Rj}''\dot{\phi}' \qquad 10.5$$

This theorem is valid provided the mechanical rotational system is invariable, contains no internal source of energy or unilateral device, linearity in the relations between torques and rotational velocities, and provided the applied torques $f_{R1}, f_{R2}, f_{R3} \ldots f_{Rn}$ are all of the same frequency.

In the simple case in which there are only two torques, as illustrated in the mechanical rotational system of Fig. 10.1, equation 10.5 becomes

$$f_R'\dot{\phi}'' = f_R''\dot{\phi}' \qquad 10.6$$

where f_R', f_R'' and $\dot{\phi}', \dot{\phi}''$ are the torques and angular velocities depicted in the mechanical rotational system of Fig. 10.1.

D. *Acoustical Reciprocity Theorem.*[2]—From equation [3] 1.4 of "Acoustical Engineering"

$$\frac{dv}{dt} + \frac{1}{\rho}\operatorname{grad} p_0 = 0 \qquad 10.7$$

[2] Rayleigh, "Theory of Sound," Vol. II, p. 145, MacMillan and Co., London, 1926.

[3] Olson, "Acoustical Engineering," D. Van Nostrand Co., Princeton, N. J., 1957.

Consider two independent sets of pressures p', p'' and particle velocities v' and v''. Multiply equation 10.4 by the p and v of the other set.

$$v'' \frac{dv'}{dt} - v' \frac{dv''}{dt} + \frac{1}{\rho} v'' \operatorname{grad} p_0' - \frac{1}{\rho} v' \operatorname{grad} p_0'' = 0 \qquad 10.8$$

If p and v vary as a harmonic of the time, equation 10.5 becomes

$$v' \operatorname{grad} p_0'' - \frac{1}{\rho} v' \operatorname{grad} p_0'' = 0 \qquad 10.9$$

There is the following relation:

$$v \operatorname{grad} p = \operatorname{div} vp - p \operatorname{div} v \qquad 10.10$$

From equations 1.9 and 1.10 of "The Elements of Acoustical Engineering"

$$\frac{1}{\gamma p_0} \frac{\partial p}{\partial t} + \operatorname{div} v = 0 \qquad 10.11$$

From equations 10.8, 10.9 and 10.10

$$\operatorname{div} (v'' p_0' - v' p_0'') = 0 \qquad 10.12$$

The relation of equation 10.12 is for a point. Integration of equation 10.12 over a region of space gives

$$\iint (v'' p_0' - v' p_0'') ds = 0 \qquad 10.13$$

If, in an acoustical system comprising a medium of uniform density and propagating irrotational vibrations of small amplitude, a pressure p' produces a particle velocity v' and a pressure p'' produces a particle velocity v'', then

$$\iint (v'' p' - v' p'')_n ds = 0 \qquad 10.14$$

where the surface integral is taken over the boundaries of the volume.

In the simple case in which there are only two pressures, as illustrated in the free field acoustical system of Fig. 10.1, equation 10.14 becomes

$$p' v'' = p'' v' \qquad 10.15$$

where p', p'' and v', v'' are the pressures and particle velocities depicted in the free field acoustical system of Fig. 10.1.

The above theorem is applicable to all acoustical problems. However, the above theorem can be restricted to lumped constants as follows: In an acoustical system composed of inertance, acoustical capacitance and acoustical resistance let a set of pressures $p_1', p_2', p_3' \ldots p_n'$ all harmonic of the same frequency acting in n points in the system, produce a volume current distribution $\dot{X}_1, \dot{X}_2, \dot{X}_3 \ldots \dot{X}_n$, and let a second set of pressures $p_1'', p_2'', p_3'' \ldots p_n''$ of the same frequency as the first, produce a second volume current distribution $\dot{X}_1'', \dot{X}_2'', \dot{X}_3'' \ldots \dot{X}_n''$. Then

$$\sum_{j=1}^{n} p_j'\dot{X}_j'' = \sum_{j=1}^{n} p_j''\dot{X}_j' \qquad 10.16$$

This theorem is valid provided the acoustical system is invariable, contains no internal source of energy or unilateral device, linearity in the relations between pressures and volume currents and complete reversibility in the elements, and provided the applied pressures p_1, p_2, $p_3 \ldots p_n$ are all of the same frequency.

In the simple case in which there are only two pressures, as illustrated in the acoustical system of lumped constants in Fig. 10.1, equation 10.16 becomes

$$p'\dot{X}'' = p''\dot{X}' \qquad 10.17$$

where p', p'' and $\dot{X}'$, $\dot{X}''$ are the pressures and volume currents depicted in the acoustical system of lumped constants in Fig. 10.1.

E. *Mechanical-Acoustical Reciprocity Theorem.*—In an interconnected mechanical-acoustical system let a set of forces $f_{M1}' \ldots f_{Mn}'$ act in the mechanical system, and a set of pressures $p_1' \ldots p_n'$ act in the acoustical system with the resultant velocities $\dot{x}_1' \ldots \dot{x}_n'$ in the mechanical system and with the resultant volume currents $\dot{X}_1' \ldots \dot{X}_n'$ in the acoustical system; let also, f'', $\dot{x}''$, p'' and $\dot{X}''$ represent a second set of such forces, velocities, pressures and volume currents. Then

$$\sum_{j=1}^{n} (f_{Mj}'\dot{x}_j'' + p_j'\dot{X}_j'') = \sum_{j=1}^{n} (f_{Mj}''\dot{x}_j' + p_j''\dot{X}_j') \qquad 10.18$$

In the simple case in which there is only one force in the mechanical system and one pressure in the acoustical system

$$f_M'\dot{x}'' = p''\dot{X}' \qquad 10.19$$

Equation 10.19 states that if a unit force f_M' in the mechanical system produces a certain volume current $\dot{X}'$ in the acoustical system, then a unit pressure p'' acting in the acoustical system will produce a velocity $\dot{x}''$ in the mechanical system which is numerically the same as the volume current previously produced in the acoustical system.

The mechanical-acoustical reciprocity theorem is illustrated in Fig. 10.2*A*.

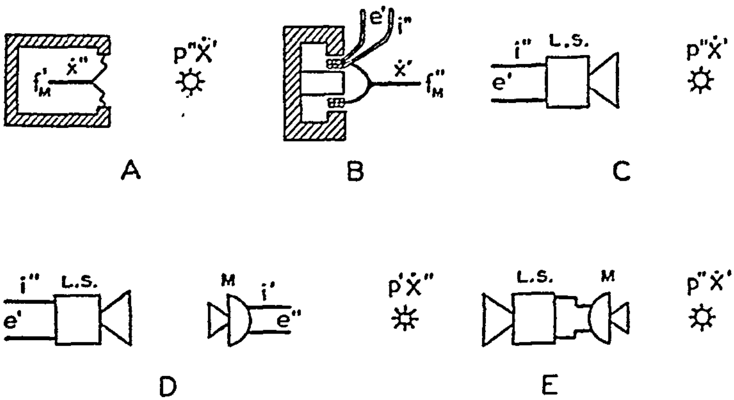

FIG. 10.2. Reciprocity in the following systems depicted above:

A. Mechanical-acoustical.
B. Electrical-mechanical.
C. Electrical-mechanical-acoustical.
D. Electrical-mechanical-acoustical-mechanical-electrical.
E. Acoustical-mechanical-electrical-mechanical-acoustical.

F. *Electrical-Mechanical Reciprocity Theorem.*—In an interconnected electrical-mechanical system let a set of electromotive forces $e_1' \ldots e_n'$ act in the electrical system, and a set of forces $f_{M1}' \ldots f_{Mn}'$ act in the mechanical system with the resultant currents $i_1' \ldots i_n'$ in the electrical system and with the resultant velocities $\dot{x}_1' \ldots \dot{x}_n'$ in the mechanical system; let also, e'', i'', f_M'' and $\dot{x}''$ represent a second set of electromotive forces, currents, forces and velocities. Then

$$\sum_{j=1}^{n} (e_j' i_j'' + f_{Mj}' \dot{x}_j'') = \sum_{j=1}^{n} (e_j'' i_j' + f_{Mj}'' x_j') \qquad 10.20$$

In the simple case in which there is only one electromotive force in the electrical system and one force in the mechanical system

$$e' i'' = f_M'' \dot{x}' \qquad 10.21$$

Equation 10.21 states that if a unit electromotive force e' in the electrical system produces a certain velocity $\dot{x}'$ in the mechanical system, then a unit force f_M'' in the mechanical system will produce a current i'' in the electrical system which is numerically the same as the velocity previously produced in the mechanical system.

The electrical-mechanical reciprocity theorem is illustrated in Fig. 10.2*B*.

G. *Electrical-Mechanical-Acoustical Reciprocity Theorem.*—Since reciprocity relations hold in electrical-mechanical and mechanical-acoustical systems, they will also hold the three systems interconnected in the order electrical, mechanical, acoustical. This type of system embraces practically all electroacoustic transducers.

For the simple case of a pressure p' in a sound field producing a current i' in the electrical system and a voltage e'' in the electrical system producing a volume current $\dot{X}''$ in the sound field the reciprocity relation may be written

$$\iint (p'\dot{X}'')ds = e''i' \qquad 10.22$$

Equation 10.23 states that if, in the electrical system of a loud speaker, a generator of electromotive force e'' produces, at a point in a sound field, a volume current $\dot{X}''$, than a numerically equal pressure in the sound field at that point will produce a current i' in the electrical system equal to the previously produced volume current $\dot{X}''$ in the sound field.

The electrical-mechanical-acoustical reciprocity theorem is illustrated in Fig. 10.2*C*.

H. *Electrical-Mechanical-Acoustical-Mechanical-Electrical Reciprocity Theorem.*—In most cases in the reproduction of sound the original sound is converted into electrical energy by a microphone which is usually an acoustical, mechanical, electrical transducer. Then it is converted back into acoustical energy by means of a loud speaker or an electrical, mechanical, acoustical transducer.

If both microphone and loud speaker are reversible and the air is the connecting medium, as shown in Fig. 10.2*D*, and if an electromotive force e' in the loud speaker will produce a current i' in the microphone, then an equal electromotive force e'' in the microphone will produce the same current i'' in the loud speaker. This may be expressed as

$$e'i'' = e''i' \qquad 10.23$$

I. *Acoustical-Mechanical-Electrical-Mechanical-Acoustical Reciprocity Theorem.*—If both microphone and loud speaker are reversible and the two connected electrically, as shown in Fig. 10.2*E*, and if a pressure p' at a point in the vicinity of the microphone will produce a volume current $\dot{X}'$ at a point in the vicinity of the loud speaker, then an equal pressure p'' at the same point in the vicinity of the loud speaker will produce the same volume current $\dot{X}''$ at the same point in the vicinity of the microphone. This may be expressed as

$$p'\dot{X}'' = p''\dot{X}' \qquad 10.24$$

10.3. Thevenin's Theorems

A. *Thevenin's Electrical Theorem.*—If an electrical impedance z_E be connected between any two points in an electrical circuit, the current i through this electrical impedance is the electromotive force e between the points prior to the connection divided by the sum of the electrical impedance z_E and the electrical impedance z_E', where z_E' is the electrical impedance of the circuit measured between the two points prior to connecting z_E.

B. *Thevenin's Mechanical Rectilineal Theorem.*—If a mechanical rectilineal impedance z_M be connected at any point in a mechanical rectilineal system, the resultant velocity of this mechanical rectilineal impedance is the product of the velocity and mechanical rectilineal impedance z_M' of the system both measured at the point prior to the connection divided by the sum of the mechanical rectilineal impedances z_M and z_M'.

C. *Thevenin's Mechanical Rotational Theorem.*—If a mechanical rotational impedance z_R be connected at any point in a mechanical rotational system, the resultant angular velocity of this mechanical rotational impedance is the product of the angular velocity and mechanical rotational impedance z_R' of the system both measured at the point prior to the connection divided by the sum of the mechanical rotational impedances z_R and z_R'.

D. *Thevenin's Acoustical Theorem.*—If an acoustical impedance z_A be connected at any point in an acoustical system, the volume current $\dot{X}$ in this acoustical impedance is the pressure p at the point prior to the connection divided by the sum of the acoustical impedance z_A and the acoustical impedance z_A', where z_A' is the acoustical impedance at the point prior to connecting z_A.

10.4. Superposition Theorem

Consider the simultaneous action of a number of electromotive forces, forces, torques or pressures distributed throughout an electrical, mechanical rectilineal, mechanical rotational or acoustical system. The current, velocity, angular velocity or volume current at any point or the electromotive force, force, torque or pressure at a location is the sum of the currents, velocities, angular velocities or volume currents or electromotive forces, forces, torques or pressures at these locations which would exist if each source were considered separately. Each source, save the one being considered, must be replaced by a unit of equivalent internal electrical, mechanical rectilineal, mechanical rotational or acoustical impedance.

10.5. Similarity Theorem

The similarity theorem states: For any given dynamical system consisting of connected particles and rigid bodies it is possible to construct another system exactly similar to it but on a different scale. If the masses and forces in the two systems bear certain ratios to each other, the performance of the two systems will be similar but with different velocities that bear a constant relationship to each other.

Specifically, to determine the relation between the various ratios involved, let the linear dimensions of the systems 1 and 2 be in the ratio $x:1$; let the masses of the corresponding particles be in the ratio $y:1$; let the rates of operation be in the ratio $z:1$, so that time between corresponding phases will be in the ratio $1:z$; and let the forces be in the ratio $w:1$. The equation of motion of a particle is given by

$$f_M = m\ddot{x} \tag{10.25}$$

where f_M = force, in dynes,
m = mass, in grams, and
$\ddot{x}$ = acceleration, in centimeters per second per second.

Referring to equation 10.25 and employing the ratios specified above, it will be seen that if m is changed in the ratio $y:1$, $\ddot{x}$ will be changed in the ratio $xz^2:1$, and f_M will be changed in the ratio $w:1$. Thus, it follows that the relation between the four numbers x, y, z and w as specified above is given by

$$w = xyz^2 \tag{10.26}$$

CHAPTER XI

APPLICATIONS

11.1. Introduction

The fundamental principles relating to electrical, mechanical rectilineal, mechanical rotational and acoustical analogies have been established in the preceding chapters. Employing these fundamental principles the vibrations produced in mechanical and acoustical systems due to impressed forces may be solved as follows: Draw the electrical network which is analogous to the problem to be solved. Solve the electrical network by conventional electrical circuit theory. Convert the electrical answer into the original system. In this procedure any problem involving vibrating systems is reduced to the solution of an electrical network. A complete treatment of the examples of the use of analogies in the solution of problems in mechanical and acoustical systems is beyond the scope of this book. However, a few typical examples described in this chapter will serve to illustrate the principles and method.

11.2. Automobile Muffler

The sound output from the exhaust of an automobile engine contains all audible frequencies in addition to frequencies below and above the audible range. The purpose of a muffler is to reduce the audible exhaust sound output. An ideal muffler should suppress all audible sound which issues from the exhaust without increasing the exhaust back pressure.

The original mufflers consisted essentially of a series of chambers which increased progressively in volume. The idea was to allow the gases to expand and thereby reduce the noise. Actually it was a series of acoustical capacitances. This muffler is quite effective. However, by the application of acoustic principles an improved muffler has been developed in which the following advantages have been obtained: smaller size, higher attenuation in the audible frequency range and reduc-

tion of engine back pressure. A cross-sectional view of the improved muffler is shown in Fig. 11.1. The acoustical network [1] shows that the system is essentially a low pass filter. The main channel is of the same diameter as the exhaust pipe. Therefore, there is no increase in acoustical impedance to direct flow as compared to a plain pipe. In order not to impair the efficiency of the engine it is important that the muffler does not increase the acoustical impedance to subaudible frequencies. The system of Fig. 11.1 can be designed so that the subaudible frequencies

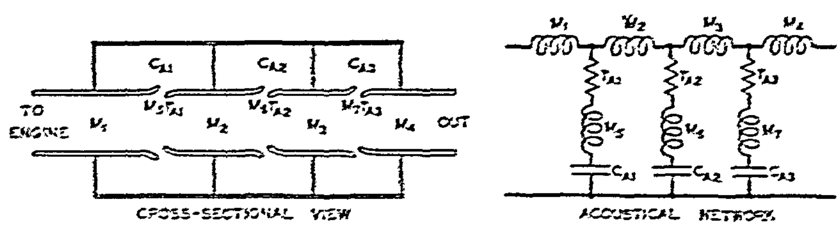

FIG. 11.1. Cross-sectional view and acoustical network of an automobile muffler.

are not attenuated and at the same time introduce high attenuation in the audible frequency range.

The terminations at the two ends of the network are not ideal. Therefore it is necessary to use shunt arms tuned to different frequencies in the low portion of the audible range. Acoustic resistance is obtained by employing slit type openings into the side chambers.

In a development of this kind the nature of sound which issues from the exhaust is usually determined. From these data the amount of suppression required in each part of the audible spectrum may be ascertained. From these data and the terminating acoustical impedances the

[1] In the illustrations in the preceding chapters the elements in the electrical network have been labeled r_E, L and C_E. However, in using analogies in actual practice the conventional procedure is to label the elements in the analogous electrical network with r_M, m and C_M for the mechanical rectilineal system, with r_R, I and C_R for a mechanical rotational system with r_A, M and C_A for an acoustical system. This procedure will be followed in this chapter in labeling the elements of the analogous electrical circuit. It is customary to label this network with the caption "analogous electrical network of the mechanical system" (or of the rotational system or of the acoustical system) or with the caption "mechanical network" (or rotational network or acoustical network). The latter convention will be used in this chapter. When there is only one path, circuit will be used instead of network.

network can be developed. In general, changes are required to compensate for approximations. In this empirical work the acoustical network serves as a guide in directing the appropriate changes.

11.3. Electric Clipper

An electric clipper is shown schematically in Fig. 11.2. The driving system is the unpolarized type described in section 8.3. The actuating

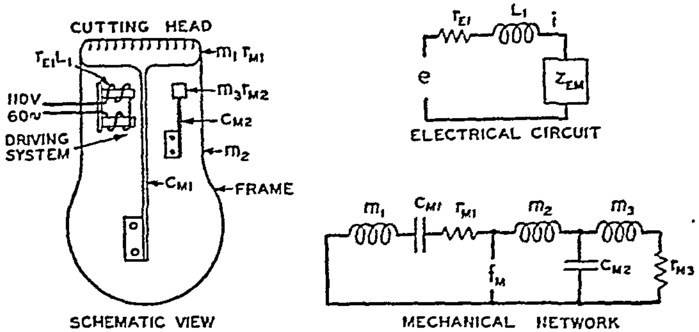

FIG. 11.2. Schematic view, electrical circuit and mechanical network of an electric clipper. In the electrical circuit: e, the alternating current line voltage. r_{E1} and L_1, the electrical resistance and inductance of the coil. z_{EM}, the motional electrical impedance of the driving system. In the mechanical circuit: f_M, the driving force. r_{M1}, m_1 and C_{M1}, the mechanical rectilineal resistance, mass and compliance of the resonant cutter. m_2, the mass of the frame. r_{M2}, m_3 and C_{M2}, the mechanical rectilineal resistance, mass and compliance of the shunt resonant system.

voltage is 110 volts 60 cycles. Therefore the frequency of the driving force is 120 cycles. This driving force acts on both the resonant clipper and the frame. The mechanical network shows that the amplitude of the frame m_2 will decrease as the mass of the frame is increased. However, the mass m_2 cannot be increased without limit because the clipper must be light so that it can be easily handled. For a suitable weight from this standpoint the vibration is too great. This vibration can be reduced by the introduction of a shunt resonant mechanical system. The mechanical rectilineal impedance of a shunt resonant mechanical circuit is very large at the resonant frequency. If this shunt mechanical circuit is tuned to 120 cycles the frame of the clipper will remain practically motionless. The mechanical network illustrates the action.

11.4. Direct Radiator Loud Speaker

The direct radiator type loud speaker shown in Fig. 11.3 is almost universally used for radio and phonograph reproduction. The mechanical circuit of this loud speaker is also shown in Fig. 11.3. The mechanical rectilineal impedance at the point f_M can be determined from the mechanical circuit. Then the motional electrical impedance can be determined from equation 8.7. The current in the voice coil can be deter-

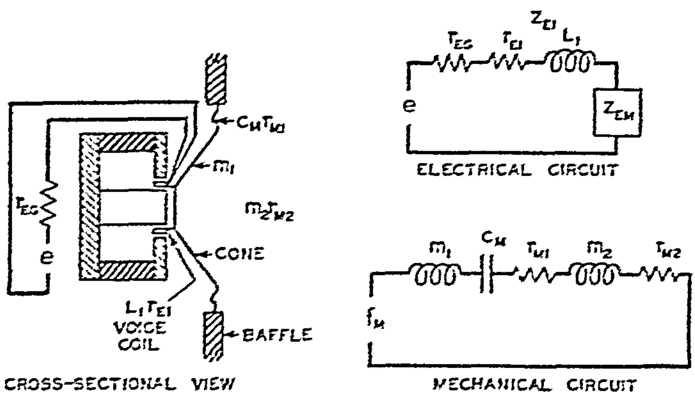

FIG. 11.3. Cross-sectional view, electrical circuit and mechanical circuit of a direct radiator dynamic loud speaker. In the electrical circuit: e, the open circuit voltage of the generator or vacuum tube. r_{EG}, the electrical resistance of the generator or vacuum tube. r_{E1} and L_1, the electrical resistance and inductance of the voice coil. z_{EM}, the motional electrical impedance of the driving system. In the mechanical circuit: m_1, the mass of the cone. r_{M1} and C_1, the mechanical rectilineal resistance and compliance of the suspension. m_2 and r_{M2}, the mass and mechanical rectilineal resistance of the air load.

mined from the electrical circuit of Fig. 11.3. The mechanical driving force can be determined from equation 8.1. The velocity can be determined from the mechanical circuit of Fig. 11.3 as follows:

$$\dot{x} = \frac{f_M}{z_{MT}} \qquad 11.1$$

where z_{MT} = total mechanical impedance at the point f_M, in mechanical ohms, and

f_M = driving force, in dynes.

The sound power output, in ergs, is given by

$$P = r_M \dot{x}^2 \qquad 11.2$$

where r_M = mechanical radiation resistance, in mechanical ohms,
$\dot{x}$ = velocity of the cone, in centimeters per second.

The object [2] is to select the constants so that the power output as given by equation 11.2 is independent of the frequency over the desired frequency range.

11.5. Rotational Vibration Damper

In reciprocating engines and other rotating machinery rotational vibrations of large amplitude occur at certain speeds. These rotational

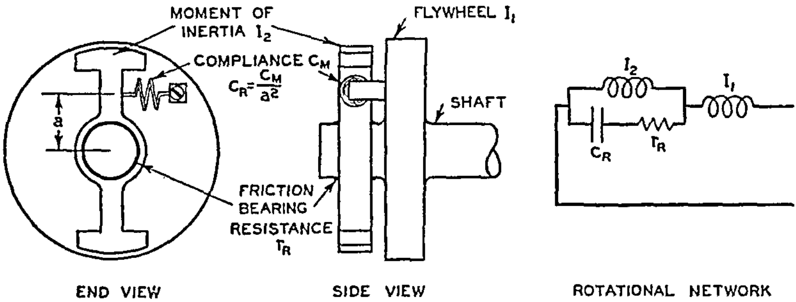

FIG. 11.4. Flywheel and vibration damper. In the mechanical network: I_1, the moment of inertia of the flywheel. I_2 the moment inertia of the damper. C_R, the rotational compliance of the damper. r_R, the mechanical rotational resistance between the damper and the shaft.

vibrations are sometimes of such high amplitude that the shafts will fail after a few hours of operation. A number of rotational vibration dampers have been developed for reducing these rotational vibrations. It is the purpose of this section to describe one of these systems for controlling the vibrations in a rotational system. The simple vibration damper, shown in Fig. 11.4, is used to control the vibrations of the fly-

[2] For a specific description of and expressions for the elements of the mechanical system see H. F. Olson, "Acoustical Engineering," D. Van Nostrand Co., Princeton, N. J., 1957. In this book all types of acoustical vibrating systems are analyzed by the use of analogies. These systems include microphones, loud speakers, phonograph pickups, telephone receivers, measuring systems, etc.

wheel. The damper consists of a moment of inertia I_2 rotating on a shaft with a mechanical rotational resistance r_R. The moment of inertia is coupled to the flywheel by a spring of compliance C_M. The rotational compliance is $C_R = C_M/a^2$, where a is the radius at the point of attachment of the spring. This system forms a shunt mechanical rotational system. The shunt mechanical rotational circuit is tuned to the frequency of the vibration. Since the mechanical rotational impedance of a shunt resonant rotational circuit is very high at the resonant frequency the angular velocity of the vibration of the flywheel will be reduced. A consideration of the mechanical rotational network illustrates the principle of the device. This is one example of the many types of vibration dampers for use in absorbing rotational vibrations. The action of these systems may also be analyzed by the use of analogies.

11.6. Machine Vibration Isolator

The vibration of a machine is transmitted from its supports to all parts of the surrounding building structure. In many instances this

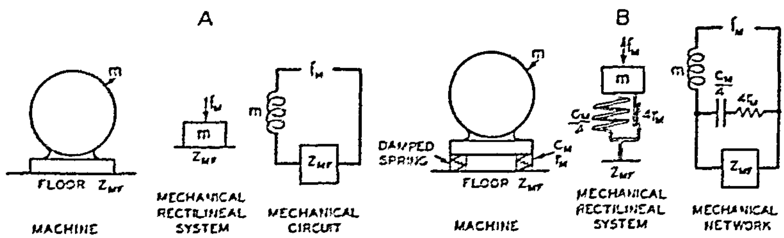

FIG. 11.5. Machine vibration isolator. A. Machine mounted directly upon the floor. In the mechanical rectilineal system and mechanical circuit: m_1, the mass of the machine. z_{MF}, the mechanical rectilineal impedance of the floor. f_M, the vibrating driving force developed by the machine. B. Machine mounted upon an isolating system. In the mechanical rectilineal system and mechanical network: m_1, the mass of the machine. $C_M/4$ and $4r_M$, the compliance and mechanical rectilineal resistance of the four damped spring mountings. z_{MF}, the mechanical rectilineal impedance of the floor. f_M, the vibrating driving force developed by the machine.

vibration may be so intense that it is intolerable. For these conditions the machine may be isolated from the base or floor upon which it is placed by introducing a mechanical isolating network.

A machine mounted directly on the floor is shown in Fig. 11.5*A*. The mechanical rectilineal system and the mechanical network for vertical vibrations is shown in Fig. 11.5*A*. The driving force f_M is due to the

vibrations of the machine. The mechanical network shows that the only isolation in the system of Fig. 11.5*A* is due to the mass of the machine.

In the simple isolating system shown in Fig. 11.5*B* the machine is mounted on damped springs. The compliance and the mechanical rectilineal resistance of the support is C_M and r_M. Since there are four supports, these values become $C_M/4$ and $4r_M$ in the mechanical rectilineal system and mechanical network for vertical vibrations. The mechanical network depicts the action of the shunt circuit in reducing the force of the vibration transmitted to the floor z_{MF}.

11.7. Mechanical Refrigerator Vibration Isolator

In the mechanical refrigerator a motor is used to drive a compressor. Since the refrigerator is a home appliance it is important that the vibra-

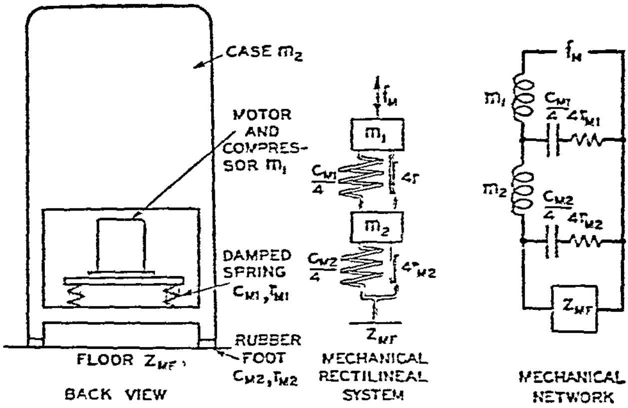

FIG. 11.6. Mechanical refrigerator vibration isolator. In the mechanical rectilineal system and the mechanical network: z_{MF}, the mechanical rectilineal impedance of the floor. $C_{M2}/4$ and $4r_{M2}$, the compliance and mechanical rectilineal resistance of the four rubber feet. m_2, the mass of the case. $C_{M1}/4$ and $4r_{M1}$, the compliance and mechanical rectilineal resistance of the four damped springs. m_1, the mass of the motor and compressor. f_M, the vibrating driving force developed by the machine.

tion and noise produced by the motor and compressor be as low in amplitude as possible. These vibrations may be suppressed by the use of an isolating mechanical network. The mechanical system, shown in Fig. 11.6, consists of the following elements: m_1, the mass of the motor and

compressor. C_{M1} and r_{M1}, the compliance and the mechanical rectilineal resistance of the springs and damping material entwined in the springs. m_2, the mass of the case. C_{M2} and r_{M2}, the compliance and the mechanical rectilineal resistance of the rubber feet. z_{MF}, the mechanical rectilineal resistance of the floor. Since there are four isolating supports for the motor and compressor and four feet on the refrigerator, the elements in the shunt circuits become $C_{M1}/4$ and $4r_{M1}$ for the isolating supports and $C_{M2}/4$ and $4r_{M2}$ for the rubber feet in the mechanical rectilineal system and the mechanical network. The mechanical network illustrates how the shunt circuit elements $C_{M1}/4$, $4r_{M1}$ and $C_{M2}/4$, $4r_{M2}$ reduce the force delivered to the floor. The shunt circuit elements $C_{M1}/4$ and $4r_{M1}$ also reduces the force delivered to the case of the refrigerator and thereby lessens the air-borne noises.

11.8. Shockproof Instrument Mounting

In order to obtain the maximum accuracy and reliability from galvanometers and other similar instruments of high sensitivity it is necessary that the mounting for the instrument be free from vibrations. Very often these instruments must be used in buildings in which the entire structure is vibrating. Any instrument support directly connected to the building will vibrate and will in turn transmit this vibration to the instrument. Under these conditions the performance of the instrument will be erratic. The instrument may be isolated from the building vibrations by means of a mechanical network of the type shown in Fig. 11.7. The instrument table legs are mounted on resiliant supports which are both a compliance C_{M1} and a resistance r_{M1}. This support reduces the vibration of the table m_1. The instrument is isolated further by the compliance C_{M2} and the mass m_2. A mechanical rectilineal resistance r_{M2} in the form of a dash pot is used to damp the vibrations of the mass m_2. The driving force at each of the four legs is f_M. Since there are four legs and four isolating supports, the elements in this shunt circuit become $C_{M1}/4$ and $4r_{M1}$ and the driving force becomes $4f_M$ in the mechanical rectilineal system and the mechanical network. The mechanical network illustrates the action of the vibrating system. The velocity of the mass m_2 is very small compared to the velocity of the floor due to the series mass elements and shunt compliance and mechanical rectilineal resistance elements. The mechanical network of Fig. 11.7 depicts the vertical modes of vibration. Of course, the system in Fig.

11.7 may vibrate in many other modes which may be solved by similar analysis but, in general, the vertical motion is the most violent and troublesome.

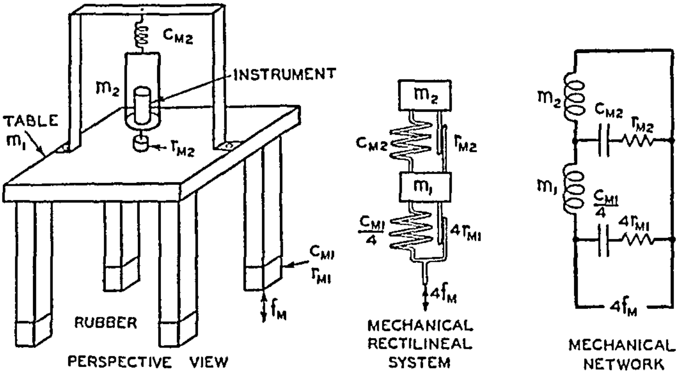

FIG. 11.7. Shockproof instrument mounting. In the mechanical rectilineal system and mechanical network: $4f_M$, the combined driving force at the four legs. $C_{M1}/4$ and $4r_{M1}$, the compliance and mechanical rectilineal resistance of the four feet. C_{M2}, the compliance of the spring suspension. r_{M2}, the mechanical rectilineal resistance of the dash pot. m_2, the mass of the instrument and carriage.

11.9. Automobile Suspension System

The riding qualities of an automobile depend primarily upon the degree of isolation of the passenger from all types of vibration. One of the principal sources of vibration is due to the uneven contour of the road over which the automobile travels. The objective of automobile designers is to reduce the vibration of the passenger to a practical minimum. A schematic view of an automobile is shown in Fig. 11.8. This system has many degrees of freedom, both rectilineal and rotational. The system depicted in the mechanical rectilineal system and mechanical network of Fig. 11.8 assumes that the forces at each of the four wheels are equal in both amplitude and phase and that vibrations occur in a vertical line. The vibrating system consists of the following elements: f_M, the driving force at each tire. C_{M1} and r_{M1}, the compliance and mechanical rectilineal resistance of the tires. m_1, the mass of the tire, wheel and axle. C_{M2}, the compliance of the spring. r_{M2}, the mechanical rectilineal resistance of the shock absorber. m_2, the mass of the frame, body, engine, etc. C_{M3} and r_{M3}, the compliance and mechanical rectilineal re-

sistance of the cushion. m_3, the mass of the passenger. Since there are four tires, wheels, springs and shock absorbers, the elements corresponding to these parts in the mechanical rectilineal system and mechanical network are as follows: $4f_M$, the driving force. $C_{M1}/4$ and $4r_{M1}$ the compliance and mechanical rectilineal resistance of the tires. $C_{M2}/4$, the compliance of the springs. $4r_{M2}$, the mechanical rectilineal resistance of the shock absorbers. However, there is no change in the case of the

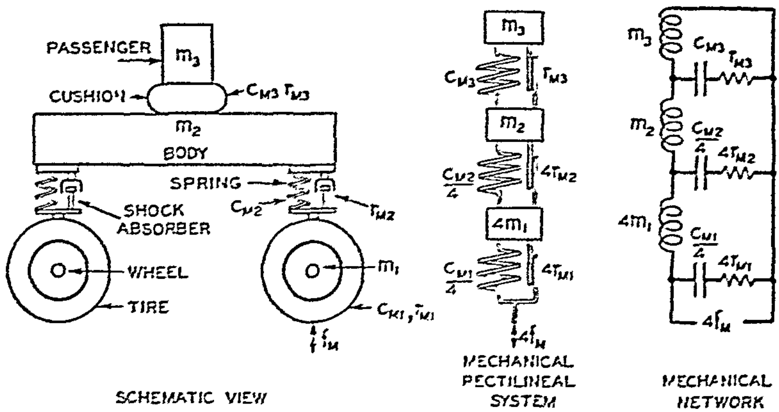

FIG. 11.8. Automobile suspension system. In the mechanical rectilineal system and mechanical network: $4f_M$, the combined driving force at the four tires. $C_{M1}/4$ and $4r_{M1}$, the compliance and mechanical rectilineal resistance of the four tires. $4m_1$, the mass of the four tires. $C_{M2}/4$, the compliance of the four springs. $4r_{M2}$, the mechanical rectilineal resistance of the four shock absorbers. m_2, the mass of the frame, body and engine. C_{M3} and r_{M3}, the compliance and mechanical rectilineal resistance of the cushion. m_3, the mass of the passenger.

m_2, C_{M3}, r_{M3} and m_3 because these are single units in the schematic view. The mechanical rectilineal resistance of the tires is quite small. The mechanical rectilineal resistance of the springs is exceedingly small. A low frequency oscillation with very little damping occurs due to the resonance of the mass m_2 of the body with the compliances C_{M2} and C_{M1} of the springs and tires. This oscillation is excited by a wavy road bed and becomes very violent when the speed divided by the wavelength corresponds to the resonant frequency. A high frequency oscillation occurs due to the resonance of the mass m_2 of the wheels and axles with the compliances C_{M1} and C_{M2} of the tires and springs. This oscillation is excited by sharp discontinuities such as cobblestones. This resonance

becomes so violent in the absence of damping that the wheels leave the road. These uncontrolled oscillations require the introduction of some form of damping for reducing the amplitude. A system has been developed in the form of the hydraulic shock absorber, which controls the oscillations. It has been found that by the use of such damping means, the compliance of the springs could be increased by use of "softer" springs and the compliance of the tires increased by the use of balloon tires. Both of these expedients have improved the riding qualities as can be seen from a consideration of the mechanical network of Fig. 11.8. A further improvement in riding qualities has been obtained by the use of better designed cushions, that is, an appropriate ratio of compliance C_{M3} to mechanical rectilineal resistance r_{M3}.

The above brief description illustrates how one of the vibration problems in an automobile may be solved by the use of analogies. As already indicated, an automobile has several modes of vibration, both rectilineal and rotational. For example, each wheel may be excited separately which may introduce a rolling, pitching or swaying motion. All of these may be analyzed by the use of analogies. The individual effects may be superposed and the gross effect of all vibrations obtained. Most of the forces, developed at the tires, are of the impulsive and not the sinusoidal type. In these cases the information on transients in electrical circuits may be applied to the mechanical system as outlined in Chapter VII.

11.10. Dynamic Microphone [3,4,5]

A cross-sectional view of a moving coil or dynamic microphone is shown in Fig. 11.9. The motion of the diaphragm is transferred to a coil located in a magnetic field. The mechanical circuit of the mechanical system consisting of the diaphragm coil and suspension system is shown in Fig. 11.9*A*.

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[3] Wente and Thuras, *Jour. Acous. Soc. Amer.*, Vol. 3, No. 1, p. 44, 1931.

[4] Wigginton, L. M., and Carroll, R. M., *Jour. Audio. Eng. Soc.*, Vol. 3, No. 2, p. 77, 1955.

[5] Olson, "Acoustical Engineering," D. Van Nostrand Co., Princeton, N. J., 1957.

where r_{M1} = mechanical resistance of the suspension system, in mechanical ohms,

m_1 = mass of the diaphragm and voice coil, in grams,

C_{M1} = compliance of the suspension system, in centimeters per dyne, and

f_M = driving force, in dynes.

The generated internal voltage e_G, in abvolts, is

$$e_G = Bl\dot{x} \tag{11.4}$$

where B = flux density in the air gap, in gausses,

l = length of the voice coil conductor, in centimeters, and

$\dot{x}$ = velocity of the voice coil, in centimeters per second.

Equation 11.4 shows that the microphone will be uniformly sensitive with respect to frequency if the velocity is independent of the frequency. The characteristics 1 and 2 in Fig. 11.9*A* were computed by employing equation 11.3, and show that a dynamic microphone, uniformly sensitive with respect to frequency, must be essentially "resistance controlled."

The characteristic marked 2, Fig. 11.9*A*, shows some falling off in velocity at the high and low frequencies. This can be corrected by the use of some additional elements (Fig. 11.9*B*). The major portion of the mechanical resistance is the silk cloth $m_2 r_{M2}$, a resistance due to the high viscosity introduced by the small holes. Slits have also been used for the resistance element.

A sectional view, the mechanical network, the electrical circuit and the response frequency characteristic of a complete dynamic microphone are shown in Fig. 11.9*B*. The mass mechanical reactance of the diaphragm is reduced at the higher frequencies by the compliance C_{M2} formed by the volume between the silk and the diaphragm. The addition of the mechanical elements C_{M2}, r_{M2} and m_2 changes the characteristic at the high frequencies from that marked 3 to that marked 5. An increase in response over an octave is obtained by the addition of these elements. A corresponding increase in response can be obtained at the low frequencies by means of the case volume C_{M3} and the addition of a tube $m_3 r_{M3}$. The addition of these elements changes the characteristic at the low frequencies from that marked 3 to that marked 4. The mechanical network shows the action of the additional elements in changing the response from the characteristic 3 to the characteristic 4-5.

11.11. Dynamic Phonograph Pickup[6,7]

A dynamic pickup is a phonograph pickup in which the output results from the motion of a conductor in a magnetic field. A dynamic pickup employing a stylus arm attached to a coil located in a magnetic field for the reproduction of lateral phonograph records is shown in Fig. 11.10.

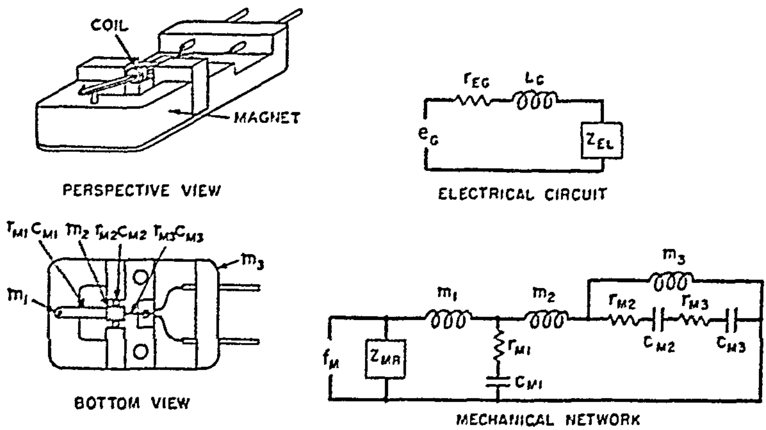

FIG. 11.10. Perspective and bottom views, mechanical network and electrical circuit of a lateral dynamic pickup. In the mechanical circuit: z_{MR}, the mechanical impedance of the record. m_1, the mass of the stylus and stylus holder. r_{M1} and C_{M1}, the mechanical resistance and compliance of the stylus arm. m_2, the mass of the coil. r_{M2} and C_{M2}, the mechanical resistance and compliance of the coil supports. r_{M3} and C_{M3}, the mechanical resistance and compliance of the longitudinal coil support. m_3, the mass of the pickup and tone arm. f_M, the force generated by velocity generator. In the electrical circuit: e_G, the open-circuit voltage developed in the coil. L_G and r_{EG}, the inductance and electrical resistance of the coil. z_{EG}, the electrical impedance of the load.

The performance of the system may be obtained from the mechanical network of Fig. 11.10. In the case of the driving force f_M the record supplies a velocity which is independent of the mechanical system. The velocity as a function of frequency of the record is the velocity frequency characteristic of the groove in the record. The only limitation is the mechanical impedance of the record z_{MR}. The open-circuit voltage of the pickup is proportional to the velocity of the coil. (Equation 11.4.)

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[6] Lindenberg, T., *Electronics*, Vol. 18, No. 6, p. 108, 1945.
[7] Olson, "Acoustical Engineering," D. Van Nostrand Co., Princeton, N. J., 1957.

The electrical generator may be considered to be the open-circuit voltage in series with the electrical impedance of the coil. (Fig. 11.10.) The coil is practically a constant electrical resistance over the audio-frequency range.

11.12. Hot-Air Heating System

A hot-air heating system consists of a blower, heater and a pipe distribution system. A schematic view of a hot-air heating system is shown in Fig. 11.11. Since the velocity of the air in the pipes is of a

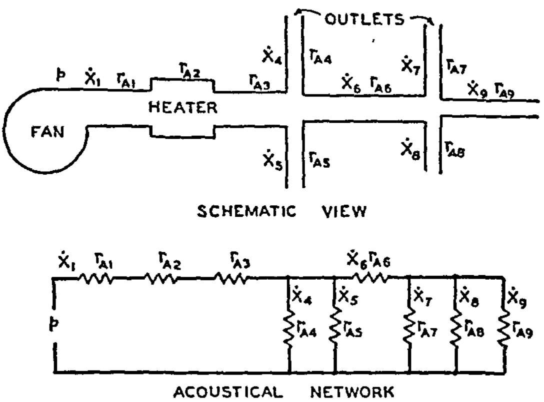


FIG. 11.11. Schematic view and acoustical network of a hot air heating system. In the acoustical network: p, the pressure delivered by the fan. $\dot{X}_1$, the volume current delivered by the fan. r_{A1}, the acoustical resistance of the pipe between the fan and the heater. r_{A2}, the acoustical resistance of the heater. r_{A3}, the acoustical resistance of the main feeder pipe between the heater and the outlets. r_{A6}, the acoustical resistance of the subfeeder pipe. $\dot{X}_6$, the volume current in the subfeeder pipe. r_{A4}, r_{A5}, r_{A7}, r_{A8} and r_{A9}, the acoustical resistances in the outlet pipes. $\dot{X}_4$, $\dot{X}_5$, $\dot{X}_7$, $\dot{X}_8$ and $\dot{X}_9$, the volume currents in the outlet pipes.

direct current type, the only elements in the system which govern the flow of air are of the resistive type. Therefore, the elements are acoustical resistances as shown in the acoustical circuit of Fig. 11.11. The flow of air or volume current in the various outlets can be determined from the acoustical resistances of the various elements and the pressure or volume current supplied by the blower.

CHAPTER XII

NOISE AND DISTORTION

12.1. Introduction

The two factors which affect the output or which limit and change the relationship between the input and output of a dynamical system are noise and distortion. Noise is a disturbance in the output of a dynamical system which does not exist in the input, and it places a lower limit upon the amplitude range of the signal or transmitted quantity. Noise is also any undesired signal or quantity, and is generated in all manner of passive and active transducers. Distortion in a dynamical system is manifested as a change in the nature and form of the output from that of the input or from that of the desired output. In general, distortion increases with the load imposed upon a dynamical system and, therefore, places an upper limit upon the output. Noise and distortion are factors of the greatest importance in communication systems, yet are also of concern in most dynamical systems. It is the purpose of this chapter to consider noise and distortion in dynamical systems.

12.2. Noise and Distortion in Machines

A dynamical system is a system involving the motion of bodies and the action of forces in producing or changing the motion of the bodies. A machine is a dynamical system consisting of two or more resistant, relatively constrained parts, which may serve to transmit and modify force and motion so as to do some desired kind of work, or a complex combination of such parts. Machines may be classified as passive and active types. A passive machine is a machine in which all the power delivered to the load is obtained from the power accepted by the machine from the source. An active machine is a machine in which the power delivered to the load is dependent upon sources of power apart from the power supplied by the outside source to the system, machine or transducer.

A complete passive machine system is schematically depicted in Fig. 12.1. It consists of the following elements: the input source, the

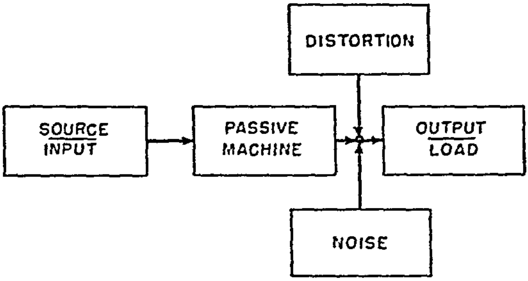


Fig. 12.1. A schematic diagram depicting the elements of a generalized passive machine system.

passive machine, the distortion source, the noise source and the output load. The output from the passive machine deviates from the input by the noise and distortion produced in the passive machine. The distortion is any departure in the form or nature of the output as contrasted with the input. Noise may be generated by the passive machine and may be classed as any erratic disturbance in the output which is not related to the input. Examples of passive machines are levers, gear trains, differentials, transmissions, mufflers, transformers, filters, loudspeakers, microphones and so forth.

A complete active machine system is schematically depicted in Fig. 12.2. It consists of the following elements: the input source, the active

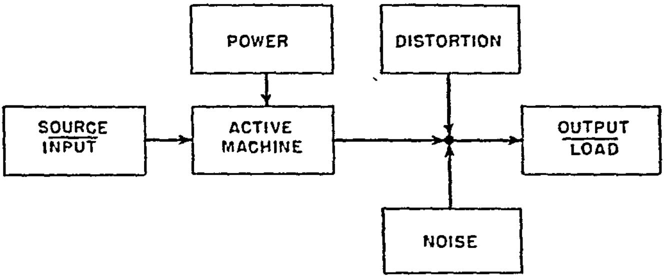


Fig. 12.2. A schematic diagram depicting the elements of a generalized active machine system.

machine such as a generator, engine or motor, the power source, the distortion source, the noise source and the output load. In general, the input source controls and determines the performance of the machine and hence the output. For the most part the power output of the active machine is derived from the power source, the output deviating from the input or the desired output by the noise and distortion produced by the active machine itself. The distortion may be manifested as variation in motion with or without changes in load. Noise may be manifested as an erratic, unrelated and unwanted disturbance in the output of the machine. Examples of active machines are vacuum tubes, transistors, controlled motors and generators, power steering, power brakes and so forth.

12.3. Noise in Dynamical Systems [1]

Noise is an erratic, intermittent or statistically random oscillation—any unwanted disturbance originating in a dynamical system. Noise usually determines the lower limit of reproduction in a sound-reproducing system. Random noises originate in all acoustical, mechanical and electrical systems. It is the purpose of this section to describe an acoustical noise due to the thermal agitation of the air molecules, a mechanical noise due to the thermal agitation of the atoms in a vibrating system and an electrical noise due to the thermal agitation of the electrons in the conductor.

A. *Noise Due to Thermal Agitation of the Air Molecules.*—Superimposed on the average atmospheric pressure are fluctuations caused by the distribution of thermal velocities of air molecules. The rms thermal sound pressure $\bar{p}$, in dynes per square centimeter, in the frequency interval between f_1 and f_2 may be obtained from the equation

$$\bar{p} = \sqrt{\int_{f_1}^{f_2} p_f^2 df} = \sqrt{\int_{f_1}^{f_2} 4kTr_A df} \qquad 12.1$$

where $p_f^2 df$ = square of the thermal acoustic pressure in the interval df,
df = differential frequency interval, in cycles per second,
r_A = acoustical radiation resistance, in acoustical ohms,
k = Boltzmann's constant, 1.37×10^{-16}, and
T = absolute temperature, in degrees Kelvin.

[1] Olson, "Acoustical Engineering," D. Van Nostrand Co., Princeton, N. J., 1957.

Equation 12.1 may be employed to calculate the noise generated in any acoustical resistance element in any acoustical system.

B. *Noise Due to Thermal Agitation of the Atoms in the Vibrating System.*—Noise is created in the acoustical resistances in a vibrating system. The effective sound pressure generated in the acoustical resistance element may be determined from equation 12.1 of the preceding section. Of course, this pressure is generated in the acoustical resistance and may be considered to be a generator in series with the acoustical resistance in the acoustical network.

In some instances it is more convenient to employ a mechanical network. In this case the rms thermal mechanical force $\bar{f}_M$, in dynes, in the frequency interval between f_1 and f_2 may be obtained from the equation

$$\bar{f}_M = \sqrt{\int_{f_1}^{f_2} f_{Mf}{}^2 df} = \sqrt{\int_{f_1}^{f_2} 4kTr_M df} \qquad 12.2$$

where $f_{Mf}{}^2 df$ = square of the thermal mechanical force in the interval df,
df = differential frequency interval, in cycles per second,
r_M = mechanical resistance of the mechanical element containing a mechanical resistance, in mechanical ohms,
k = Boltzmann's constant, 1.37×10^{-16}, and
T = absolute temperature, in degrees Kelvin.

Equation 12.2 may be employed to calculate the noise element in the mechanical resistance element in any mechanical system.

C. *Noise Due to Thermal Agitation of the Electrons in a Conductor.*—The thermal agitation of the electrons in the conductor of the electrical system of a microphone generates a fluctuating voltage.[2,3] The voltage e, in abvolts, due to the thermal agitation of the electrons in a conductor is given by

$$e = \sqrt{4kT(f_2 - f_1)r_E} \qquad 12.3$$

where k = Boltzmann's constant, 1.37×10^{-16},
T = absolute temperature, in degrees Kelvin,
$f_2 - f_1$ = width of the frequency band, in cycles per second, and
r_E = electrical resistance of the conductor, in abohms.

[2] Johnson, J. B., *Phys. Rev.*, Vol. 32, No. 1, p. 97, 1928.
[3] Nyquist, H., *Phys. Rev.*, Vol. 32, No. 1, p. 110, 1928.

Equation 12.3 may be employed to calculate the noise generated in the electrical resistance in any electrical system.

12.4. Distortion in Dynamical Systems

Distortion is a change in the output of a dynamical system from that of the desired output. Noise and certain desired changes in the output are not classed as distortion. Distortion occurs in all dynamical systems because of the nonlinear character of all types of machines and transducers, all elements being nonlinear to some extent. In general, the operation of the system may be adjusted so that the nonlinear distortion will be tolerable or acceptable. Distortion also occurs in dynamical systems because of the undesired response characteristic of the elements. This may be manifested as variations in the amplitude, velocity or acceleration of the output as a function of the frequency. Then, too, distortion occurs in dynamical systems because of transient variations. Transient distortion in a dynamical system relates to a sudden discrepancy in the output as contrasted to the input, or a reference of response or a desired response. Transient distortion usually occurs during a period of change in the output of the machine. Delay or phase distortion in a dynamical system relates to a discrepancy between the time of the response of the output as contrasted to the input, or a reference of response or a desired response.

12.5. Noise and Distortion in Sound-Reproducing Systems[4]

A complete sound-reproducing system may be represented as shown in Fig. 12.3. The first element in the chain is the information source which produces the original sound that is sent to a recorder or a transmitter. The original sound contains the ambient noise which occurs in the local environment of the source of sound. The original sound is recorded or transmitted by means of a recorder or a transmitter, which adds the inherent noise in these systems to the signal that is recorded or transmitted. In the case of a recording system there will be a storage medium, and in the case of a transmitting system there will be a transmission medium. Both of these mediums add noise inherent in their elements. The signal is reproduced at the ultimate destination by a reproducer or a receiver, which also adds noise. The final link in the

[4] Olson, "Acoustical Engineering," D. Van Nostrand Co., Princeton, N. J., 1957.

chain is the information destination. In a sound-reproducing system the ultimate useful destination of all reproduced sound is the human ear.

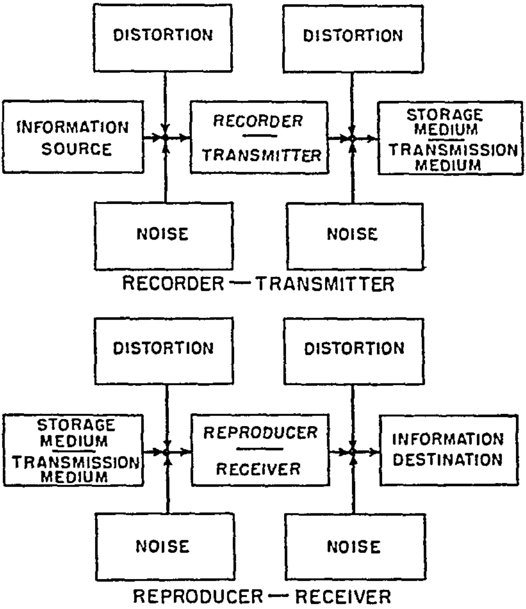


Fig. 12.3. A schematic diagram depicting the elements of a generalized communication system consisting of a transmitting or a recording section and a receiving or a reproducing section.

Referring to the system of Fig. 12.3, it will be seen that there are two important parameters involved in the transmission, namely, signal and noise. The capacity [5] C for the transmission of information of a sound-reproducing system is given by

$$C = W \log \frac{P + N}{N} \tag{12.4}$$

where W = frequency bandwidth of the transmission system, in cycles per second,
P = power of the signal, in watts, and
N = power of the noise (white noise), in watts.

[5] Shannon and Weaver, "The Mathematical Theory of Communication," The University of Illinois Press, Urbana, Illinois, 1949.

viation from a linear phase shift with frequency is a measure of the phase distortion.

The transient response characteristic provides data on the response of a system to a sudden change in the input. The deviation in the envelope of the wave output to the envelope of wave input is a measure of the transient distortion. The subject of transient response is treated in considerable detail in Chapter VII on Transients.

CHAPTER XIII

FEEDBACK

13.1. Introduction

The control of all manner of electrical, mechanical and acoustical dynamical systems plays an important function in the operation of these same systems. The term *feedback* was introduced more than two decades ago to describe generically various means for the control of dynamical systems. Specifically, feedback is used to designate a system in which a part of the output energy is referenced and fed back into the input and thereby provides a control upon the output. There are three general fields of the application of feedback, namely, regulators, servos, and electrical and electronic systems alone and in combination with mechanical systems. Although feedback control systems have been used for many years, from the foregoing it may be concluded that the concept of feedback is becoming increasingly significant in all walks of life today. It is the purpose of this chapter to describe some typical dynamical systems employing the application of feedback in various forms.

13.2. Feedback Control System [1]

A feedback control system is a control system which tends to maintain a prescribed relationship of one system variable to another system variable by comparing functions of these variables and using the difference as a means of control.

A schematic of a generalized feedback system is shown in Fig. 13.1. The elements, functions and variables in this system will now be described.

The command is the input which is established by some means external to and independent of the feedback control system.

[1] *Elec. Eng.*, Vol. 70, No. 10, p. 905, 1951.

The reference input system is a system which converts the command to the reference input.

The reference input is a variable established as a standard of comparison for a feedback control system by virtue of its relation to the command.

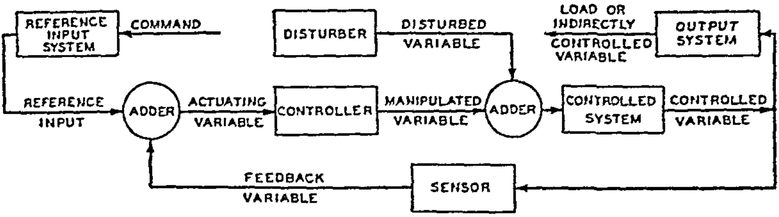


FIG. 13.1. A schematic diagram depicting the elements and functions of a feedback system.

An adder is a system which adds and combines two inputs into a single output.

The controlled variable is that quantity of the controlled system which is directly measured or controlled.

The feedback variable is that quantity which is a function of the controlled variable and which is compared to the reference input to obtain the actuating variable.

The sensor comprises the portion of the feedback control system which establishes the relationship between the feedback variable and the controlled variable.

The actuating variable is the sum of the reference input and the feedback variable.

The controller comprises the portion of the feedback control system which is required to produce the manipulated variable from the actuating variable.

The controlling system consists of the combination of the sensor and controller.

The manipulated variable is that quantity or condition which the controller applies to the controlled system.

The disturber is the system that produces the disturbed variable.

The disturbed variable is that quantity which tends to affect the value of the controlled variable.

The controlled system is the body, process or machine, a particular quantity or condition which is to be controlled.

The output system is the body, process or machine which determines the relationship between the indirectly controlled variable and the controlled variable.

The indirectly controlled variable, or load, is that quantity or condition which is controlled by virtue of its relation to the controlled variable and which is not directly measured for control.

The system shown in Fig. 13.1 is a generalized system. There are many variations of this system; these may be either simpler or more complex depending upon the application.

13.3. Action of a Feedback System [2]

The term *feedback system* is used to designate an arrangement of elements of a dynamical system in which a part of the output energy is added to the input energy and fed back into the input, thereby providing a control upon the output. The elements of a simple feedback system are shown in the schematic diagram of Fig. 13.2. The system

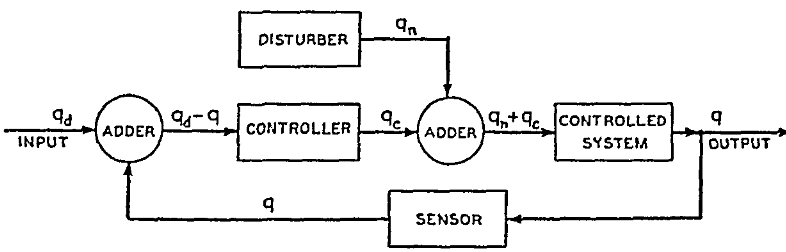


FIG. 13.2. A schematic diagram of a feedback system. q, the controlled output. q_c, the referenced input to the controlled system. q_d, the reference input. q_n, the noise or disturbance input.

is arranged so that the energy input serves as the reference to the regulated energy output. The reference input is combined with the controlled output of the controlled system, sent through the controller and then fed to the input of the controlled system. In any feedback system there are variations in the relationship between the reference input and the regulated output. There are variations which are of a short duration as the system is adjusting itself to variations in the load or in the reference input; there are also variations due to noise originating in various

[2] Trimmer, "Response of Physical Systems," John Wiley and Sons, New York, N. Y., 1950.

parts of the system. The disturbances can be represented as originating in the disturber shown in Fig. 13.2; the output of the disturber is fed to the controlled system.

The performance of the controlled system, considered to be a first-order system, can be expressed by a first-order differential equation as follows:

$$\tau\dot{q} + q = q_c + q_n \qquad 13.1$$

where q = controlled output variable, which may represent amplitude, velocity, pressure, torque and so forth,

τ = time constant of the system. (The time constant is the time in seconds it takes for the variable output q to increase to a certain fraction of its ultimate value when an input is applied to the controlled system. See also "relaxation time" on page 16.)

$\dot{q}$ = controlled output, differentiated with respect to time,

q_c = referenced input to the controlled system, and

q_n = noise or disturbance input to the controlled system.

The solution of the differential equation 13.1 is given by

$$q = (q_n + q_c)(1 - \epsilon^{-\frac{t}{\tau}}) \qquad 13.2$$

where t = time, in seconds.

Equation 13.2 gives the response of the controlled system without feedback. The next consideration will show the effect of introducing feedback upon the response of the controlled system.

The output of the controlling system, that is, the combination of the sensor and the input, is given by

$$q_c = K(q_d - q) \qquad 13.3$$

where q_d = reference input, and

K = positive constant of the controlling system.

Combining equations 13.1 and 13.3, the result may be written

$$\tau\dot{q} + q = K(q_d - q) + q_n \qquad 13.4$$

Equation 13.4 may be rearranged as follows:

$$\frac{\tau}{1 + K}\dot{q} + q = \frac{K}{1 + K}q_d + \frac{q_n}{1 + K} \qquad 13.5$$

The solution of the differential equation 13.5 is given by

$$q = \left(\frac{Kq_d + q_n}{1 + K}\right)(1 - \epsilon^{-\frac{(1+K)t}{\tau}}) \qquad 13.6$$

Equation 13.6 gives the response of the controlled system with feedback. The effect of feedback may be deduced from equation 13.6 as follows: The time constant of the system is reduced, and the noise in the controlled output is reduced. These two effects are desirable. There is a slight change in the value of the reference input variable q_d, which may be considered to be undesirable. The latter undesirable characteristic can be eliminated by arranging the output of the controlling system so that the operating conditions will satisfy the following equation:

$$q_c = K(q_d - q) + q_d \qquad 13.7$$

The equation for the system obtained by combining equations 13.1 and 13.7 becomes

$$\tau\dot{q} + q = K(q_d - q) + q_d + q_n \qquad 13.8$$

Equation 13.8 may be rearranged as follows:

$$\frac{\tau}{1 + K}\dot{q} + q = q_d + \frac{q_n}{1 + K} \qquad 13.9$$

The solution of the differential equation 13.9 is given by

$$q = \left(q_d + \frac{q_n}{1 + K}\right)(1 - \epsilon^{-\frac{(1+K)t}{\tau}}) \qquad 13.10$$

Equation 13.10 gives the response of the controlled system with feedback.

By comparing equations 13.2 and 13.10 for the response with and without feedback, it will be seen that as the value of K is increased, the time constant of the system is reduced and the noise in the controlled output is reduced. As a consequence, the relation between the input q_d and the output q becomes closer as the value of K is increased.

13.4. Hydraulic Regulator

One of the simplest and oldest regulating, or feedback, systems is the water-tank level regulator shown in Fig. 13.3. The incoming water

supply is controlled by a valve which in turn is actuated by the float. If the water level falls, the valve opens and the tank begins to fill. This process continues until a level of the water is attained so that the float and lever close the valve. In this process the level of the water in the tank is maintained within rather narrow limits and insures that the tank is always filled.

The schematic diagram of Fig. 13.3, which is the acoustical network

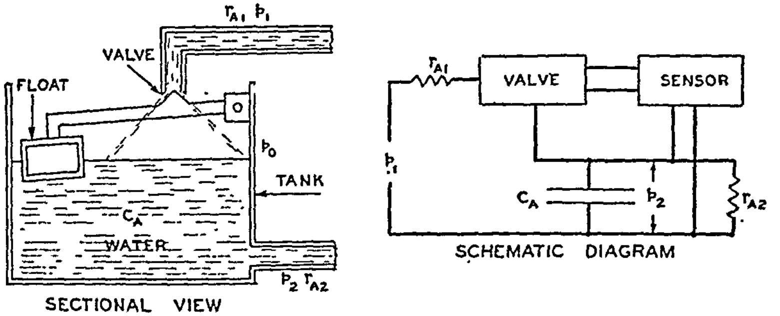


FIG. 13.3. A sectional view and the schematic diagram of a hydraulic regulator. p_1, the input pressure. r_{A1}, the acoustical resistance of the input system. C_A, the acoustical capacitance of the tank. $p_0 = 0$, the desired water level. p_2, the output pressure. r_{A2}, the acoustical resistance of the output system.

of the hydraulic system, shows that the hydraulic regulator is a feedback system in which the float is the sensor which operates the valve. The valve and the sensor constitute the controlling system; the capacitance constitutes the controlled system. The reference p_0 for desired water level leads to the the pressure p_2 in the output, which is proportional to the height of the liquid in the tank or the pressure in the acoustical capacitance C_A.

13.5. Engine Governor

An old example of a control, or feedback, system is Watt's centrifugal, or flyball, governor operating upon an engine or turbine. A perspective view of a steam turbine equipped with a governor is shown in Fig. 13.4. A spring is arranged to pull the balls of the governor towards the shaft. The centrifugal force acts to pull the balls away from the shaft. Thus, it will be seen that for each speed there will be a certain deflection of the balls from the shaft—the higher the speed the greater the deflection.

The deflection of the balls is used to control a valve admitting steam to the turbine. The amount of steam fed to the turbine varies inversely

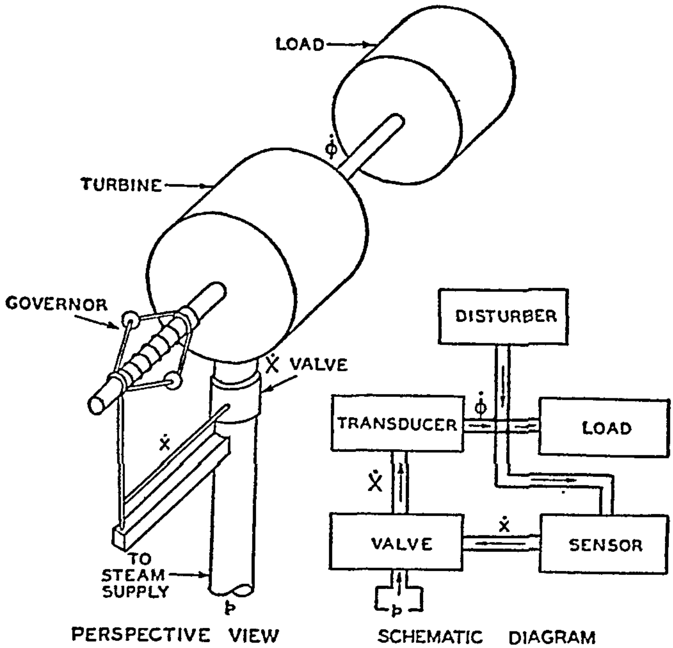


FIG. 13.4. A perspective view of a steam turbine, governor and load, and a schematic diagram of the system. p, the input steam pressure. $\dot{X}$, the volume velocity of the steam input. $\dot{x}$, the velocity of the valve. $\dot{\phi}$, the rotational velocity of the turbine shaft.

with the deflection. Thus, it will be seen that the governor acts to maintain constant speeds.

The schematic diagram of Fig. 13.4 shows that the flyball governor is a feedback system. The sensor on the output is the governor. The sensor and the valve constitute the controlling system. The turbine converts the steam flow input to mechanical rotational output and is termed the transducer. The turbine output is connected to a load. Disturbances in the system due to variations in the load, variations in the steam pressure and so forth, are depicted as due to a disturber. The controlling system consists of the sensor and valve and the controlled system consists of the transducer and the load. Thus, it will be seen that the system of Fig. 13.4 is a feedback system.

13.6. Power Steering

A recent and widespread application of a mechanical feedback system is the automobile power-steering system. A sectional view and schematic diagram of a power-steering system are shown in Fig. 13.5. The

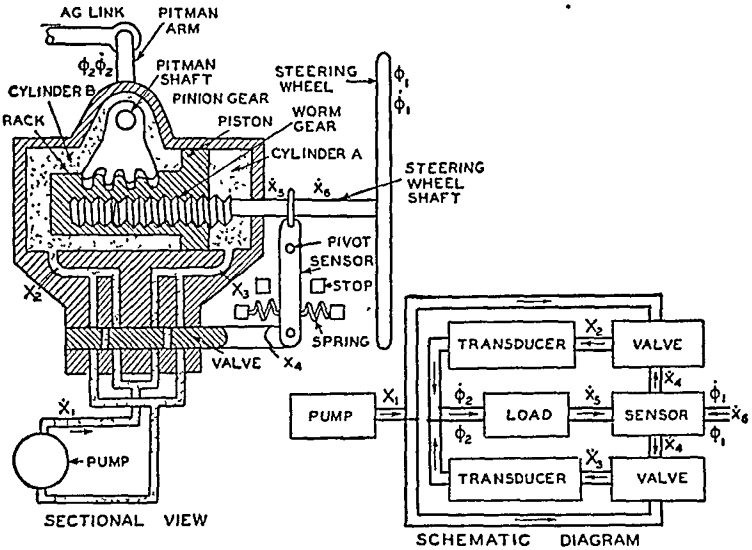


FIG. 13.5. A combination sectional and side view and schematic diagram of a power steering mechanism. ϕ_1, the rotational position of the steering wheel. $\dot{\phi}_1$, the rotational velocity of the steering wheel. $\dot{X}_1$, the volume current flow of the oil from the pump. $\dot{X}_2$ and $\dot{X}_3$, the volume current flow to the transducer. ϕ_2, the rotational position of the drag link, or load. $\dot{\phi}_2$, the angular velocity of the drag link, or load. $\dot{x}_4$, the velocity imparted by the sensor to the valve. $\dot{x}_5$ and $\dot{x}_6$, the velocities produced by the thrust of the steering wheel shaft by the drag link, or load, and the steering wheel.

reference input derives from the rotational position or rotational velocity of the steering wheel. The controlling system consists of the sensor and valve. The controlled system consists of the pump, piston, rack and pinion gear. The combination of the piston, rack and pinion gear is termed the transducer, which converts the energy in the fluid flow from the pump to mechanical rotational motion.

The operation of the system may be described as follows: Suppose the steering wheel is turned so that the steering wheel shaft is exerting

an axial force on the piston to move the piston to the left. Under this condition the axial reaction or thrust by the worm gear of the steering wheel shaft moves the shaft to the right. The axial motion of the steering wheel shaft is transmitted to the sensor. Under the action of the steering wheel shaft thrust, the sensor will move the valve to the left and thereby admit fluid under high pressure output from the pump to cylinder *A*, at the same time admitting fluid from cylinder *B* to the low pressure input to the pump. Under these conditions, because of a difference in pressure on the two sides of the piston, the piston will move to the left until there is no longer any thrust on the sensor, in which case the valves are closed to both cylinders. It will be seen that the force required at the steering wheel to operate the mechanism is only that required to deflect the springs of the sensor and to overcome friction in the steering wheel shaft. The schematic diagram shows that the power-steering system is a feedback system. The rotational position ϕ_1 or the rotational velocity $\dot{\phi}_1$ of the steering wheel is the reference input. The sensor and the valve constitute the controlling system. The controlled system consists of the transducer, which converts the fluid flow $\dot{X}_1$ from the pump to the rotational position ϕ_2 or the rotational velocity $\dot{\phi}_2$ of the drag link, or the load.

13.7. Electronic Feedback Amplifier[2]

A. *Action of the Electronic Feedback Amplifier.*—The electronic feedback amplifier consists of an electronic amplifier and a feedback loop

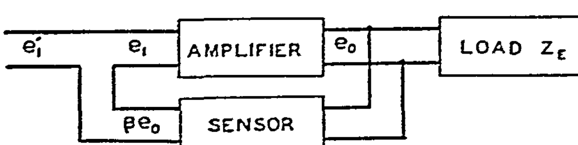


FIG. 13.6. A schematic diagram of an electronic feedback amplifier. e_1', the input voltage. e_1, the input voltage to the amplifier. e_0, the output voltage from the amplifier. βe_0, the input feedback voltage. z_E, the electrical impedance of the load.

arranged so that a voltage derived from the output is fed in opposition to the applied signal input to the amplifier. A schematic diagram of a feedback amplifier is shown in Fig. 13.6.

[2] Bode, "Network Analysis and Feedback Amplifier Design," D. Van Nostrand Co., Princeton, N. J., 1945.

The voltage amplification of the amplifier is given by

$$A = \frac{e_0}{e_1} \qquad 13.11$$

where A = voltage amplification of the amplifier,
e_0 = output voltage of the amplifier, and
e_1 = input voltage to the amplifier.

The input voltage to the feedback system is

$$e_1' = e_1 - \beta e_0 \qquad 13.12$$

where e_1' = input voltage to the feedback system, and
β = gain constant of the sensor.

The voltage amplification of the feedback system is given by

$$A_F = \frac{e_0}{e_1'} \qquad 13.13$$

where A_F = voltage amplification with feedback.

Combining equations 13.11, 13.12 and 13.13,

$$A_F = \frac{e_0}{e_1 - \beta e_0} = \frac{\dfrac{e_0}{e_1}}{1 - \beta\dfrac{e_0}{e_1}} = \frac{A}{1 - \beta A} \qquad 13.14$$

The quantity βA is termed the feedback factor or feedback loop amplification. Equation 13.14 may be written

$$A_F = -\frac{1}{\beta}\left(\frac{1}{1 - \dfrac{1}{A\beta}}\right) \qquad 13.15$$

Equation 13.15 shows that for large values of $A\beta$ the amplification becomes the reciprocal of the fraction of the output that is superimposed upon the amplifier input and is practically independent of the characteristics of the amplifier. This is characteristic of a typical feedback system. See Sect. 13.3. If the sensor or feedback loop introduces no phase shift,

the amplification will be independent of the frequency and dependent upon the magnitude of βA.

B. *Nonlinear Distortion Reduction in the Electronic Feedback Amplifier.*—The feedback amplifier reduces the nonlinear distortion which originates in the amplifier. The ratio of the nonlinear distortion with feedback to that without feedback is given by

$$\frac{D_F}{D} = \frac{1}{1 - A\beta} \qquad 13.16$$

where D_F = nonlinear distortion with feedback,
D = nonlinear distortion without feedback,
A = amplifier gain without feedback, and
β = gain constant of the sensor.

Referring to equation 13.16, it will be seen that if $A\beta$ is made large, which means a large amount of feedback, there will be a great reduction in nonlinear distortion.

C. *Noise Reduction in the Electronic Feedback Amplifier.*—It was shown in Sect. 13.3 that a feedback system will reduce disturbances, or noise. The ratio of the noise to signal with feedback to that without feedback is given by

$$\frac{S_F}{S} = \frac{A_{F0}}{A_0(1 - A\beta)} \qquad 13.17$$

where S_F = noise to signal ratio with feedback,
S = noise to signal ratio without feedback,
A_{F0} = amplification between the point 0 in the amplifier in which the noise is introduced and the output with feedback,
A_0 = amplification between the point 0 in the amplifier in which noise is introduced and the output without feedback, and
β = gain constant of the sensor.

In the use of equation 13.17, it is assumed that the output voltages are the same with and without feedback and that the introduced noise is the same with or without feedback. Referring to equation 13.17, it will be seen that the use of feedback will effect a considerable reduction in noise.

D. *Stability of the Electronic Feedback Amplifier.*[4,5]—There is another factor in feedback systems, namely, stability. Equation 13.14 gives the amplification of a feedback amplifier as follows:

$$A_F = \frac{A}{1 - \beta A} \qquad 13.18$$

βA is termed the loop amplification. If the real part of βA is negative, there will be negative feedback with decrease in amplification. If the real part of βA is positive and less than 1, there will be positive feedback with an increase in amplification. If $\beta A \geqq 1$, the amplification is infinite and the system is unstable.

In general, βA is real and negative for the midfrequency range of a feedback amplifier. However, β and A are vector quantities and have

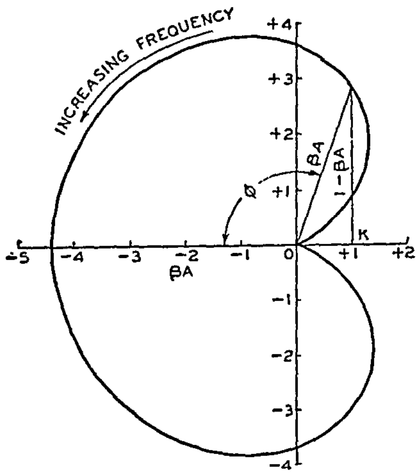


FIG. 13.7. Locus of the βA vectors for a two stage resistance capacitance coupled amplifier. The polar graph is termed the Nyquist diagram.

different magnitudes and phase angles at the low and high frequencies than in the midfrequency range.

The conditions for stability have been outlined by Nyquist and may be described by means of the Nyquist diagram. Fig. 13.7.

[4] Nyquist, H., *Bell System Tech. Jour.*, Vol. 11, No. 1, p. 128, 1932.
[5] Black, H. S., *Bell System Tech. Jour.*, Vol. 13, No. 1, p. 1, 1934.

To find the magnitude and phase of βA at any frequency, draw a circle of radius A_F/A with the center at K and a second circle of radius $|\beta A|$ with the center at 0. The line from 0 to the point of intersection of the two circles is the complex number βA. The distance from any point K is equal to $(1 - \beta A)$, where β is negative for negative feedback. The locus of βA is plotted from low to high frequencies which results in a polar diagram. ϕ is the angle between the vector βA and the absicissa. It will be seen that βA has considerable magnitude for ϕ greater than 90° but is zero for $\phi = 180°$. βA is negative, real and of the maximum value, for $\phi = 0$. For an amplifier to be stable, the locus of βA must not include the point K (1,0).

13.8. Microphone Calibrating System

In carrying out research and development work on microphones, a loudspeaker which delivers constant sound pressure over the range from 20 to 20,000 cycles per second is an indispensable part of the test equipment. Although relatively uniform response can be obtained from loudspeakers, the variations are still too large for use in obtaining accurate response frequency characteristics of microphones. For this application, feedback can be used to reduce the variations in the output

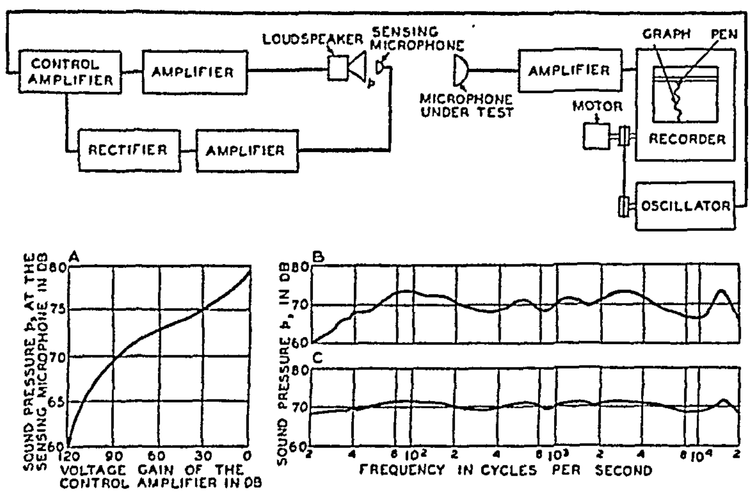


FIG. 13.8. A schematic diagram of a microphone calibrating system employing a feedback control system for reducing the variations in the sound pressure output of the loudspeaker. A. The characteristics of the control system. B. Sound pressure output without control. C. Sound pressure output with control.

of the loudspeaker as shown in Fig. 13.8. A sensing microphone with a uniform response frequency characteristic is placed in front of the loudspeaker. The output of the microphone is amplified, rectified and fed to the control grids of a variable control amplifier. The signal output of the oscillator is fed to the input of the control amplifier. The amplification characteristic of the control amplifier as a function of the sound pressure at the sensing microphone is shown in Fig. 13.8A. It will be seen that amplification of the control amplifier is an inverse function of the sound pressure of the sensing microphone. Under these conditions the combination of the sensing microphone and control amplifier operates to maintain constant sound pressure at the sensing microphone within the limitations of the feedback system. The sound pressure output of the loudspeaker without and with the feedback system is given by curves B and C of the graph of Fig. 13.8. It will be seen that variations in response are reduced by the use of the feedback system.

13.9. Constant Speed System [6]

For certain applications, as for example, recording and reproducing television signals by means of magnetic tape, constant speed of the tape is required. A system for obtaining constant rotational speed is shown in Fig. 13.9. The capstan and tape system is driven by an induction motor. The alternator and brake are connected to the same shaft. The phase of the output signal of the alternator is compared with the phase of a constant frequency signal from an electronic generator. The difference in the two signals is rectified, amplified and fed to the brake. The frequency of the oscillator is selected so that with the feedback system operating, the motor speed is reduced about 3 per cent. When the phase angle between the alternator and the oscillator is large, the brake current is small, which circumstance increases the speed of rotation and thereby serves to decrease the phase between alternator and the oscillator. Fig. 13.9*A*. When the phase angle between the alternator and oscillator is small, the brake current is large, which circumstance decreases the speed of rotation and thereby serves to increase the phase between the alternator and the oscillator. Fig. 13.9*B*. Thus, it will be seen that the phase difference between the output alternator and oscillator will be maintained at practically constant value, a fact

[6] Morgan and Artzt, *RCA Review*, Vol. 17, No. 3, p. 350, 1956.

which means that the instantaneous rotational speed of the shaft will be maintained at a practically constant value. The average rotational velocity will be maintained as absolutely constant as the reference

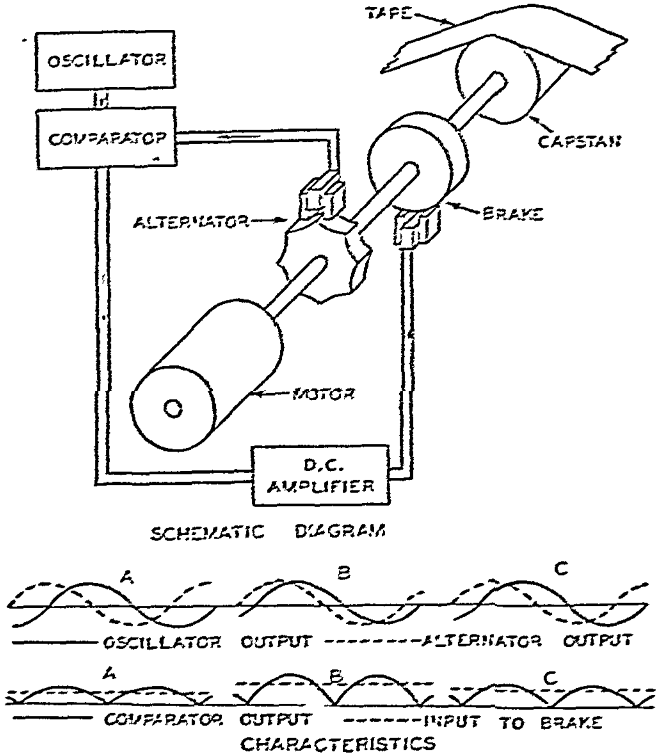


Fig. 13.9. A perspective view of a constant speed system employing an alternator, oscillator, comparator and brake. The wave characteristics depict the oscillator, alternator and comparator outputs, and the input to the brake.

frequency. The average operating point for a certain set of conditions is depicted in Fig. 13.9C.

13.10. Feedback Cutter

During the past decade, feedback has been used on a large scale to control all manner of vibrating systems employed in sound reproduction.

An example is the feedback cutter employed in cutting the original lacquer master in disk phonograph recording.

A sectional view, the mechanical circuit and the electrical system of a feedback lateral-type phonograph cutter[7,8] is shown in Fig. 13.10.

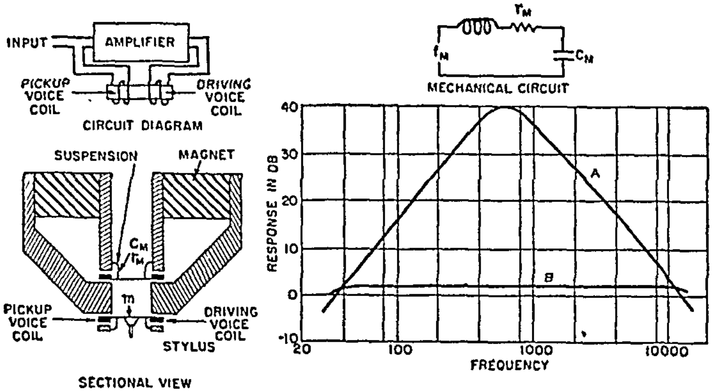


FIG. 13.10. Sectional view, mechanical circuit, electrical system and velocity response frequency characteristic of a feedback lateral-type phonograph cutter. In the mechanical circuit: f_M, the mechanical driving force. m, r_M and C_M, the mass, mechanical resistance and compliance of the vibrating system. In the graph: *A*, the velocity frequency response characteristic without feedback. *B*, the velocity response frequency characteristic with feedback.

The vibrating system is of the dynamic type with two voice coils. The vibrating system is designed so that there is a single degree of freedom over the operating frequency range. The velocity response frequency characteristic of the vibrating system shows that it is a system of one degree of freedom from 30 cycles to 16,000 cycles with the fundamental resonant frequency at 700 cycles. The output of the sensing coil is fed to the input of the amplifier, the output of the amplifier to a driving coil in an out-of-phase relationship. The signal is fed to the input of the amplifier. With the feedback in operation the velocity of the vibrating system is practically independent of the frequency over the frequency range from 30 to 16,000 cycles. The input to the amplifier can be compensated to provide the desired recording characteristic.

[7] Davis, C. C., *Jour. Audio Eng. Soc.*, Vol. 2, No. 4, p. 228, 1954.
[8] Morgan, A. R., Unpublished Report.

13.11. Feedback Pickup [9]

Feedback may be used in electrical and electromechanical systems to change such factors as the transmission and distortion characteristics, the terminal impedances and the noise levels of the systems. Feedback has been applied to cutters, calibrators and other electroacoustic devices. However, the application of feedback to phonograph pickups is a recent development. The problem in the phonograph pickup is to reduce the mechanical impedance at the stylus of the pickup so that the load presented by the stylus to the record will be reduced. The feedback phonograph pickup shown in Fig. 13.11 employs two electromechanical transducers in the feedback loop. The system for driving the stylus is an electromagnetic transducer in which the stylus is attached to the armature. The sensing and reporducing system is a ceramic transducer consisting of two barium titanate strips attached to the two sides of the steel armature. The electrical diagram of the feedback phonograph pickup is shown in Fig. 13.11; the performance of the system may be deduced from the mechanical network of Fig. 13.11. In the ceramic transducer the open-circuit voltage e, in volts, is given by

$$e = K_B x \qquad 13.19$$

where x = amplitude of the vibration of the transducer, and
K_B = constant of the system involving the material and construction of the transducer.

In the electromagnetic transducer the force f_{M2}, in dynes, produced by a current i, in abamperes, in the coil is given by

$$f_{M2} = K_I i \qquad 13.20$$

where K_I = constant involving the parameters of the electromagnetic transducer.

The problem is to adjust the amplitude and phase of the system so that a maximum displacement will be produced in C_{M4} for a minimum force f_{M1} at the stylus. The performance of the system with and without feedback is shown in Fig. 13.11. It will be seen that a tremendous reduction in the stylus force f_{M1} is obtained with feedback. This in

[9] Halter, J. B., Unpublished Report.

turn means that there will be a corresponding reduction in the force required to drive the stylus.

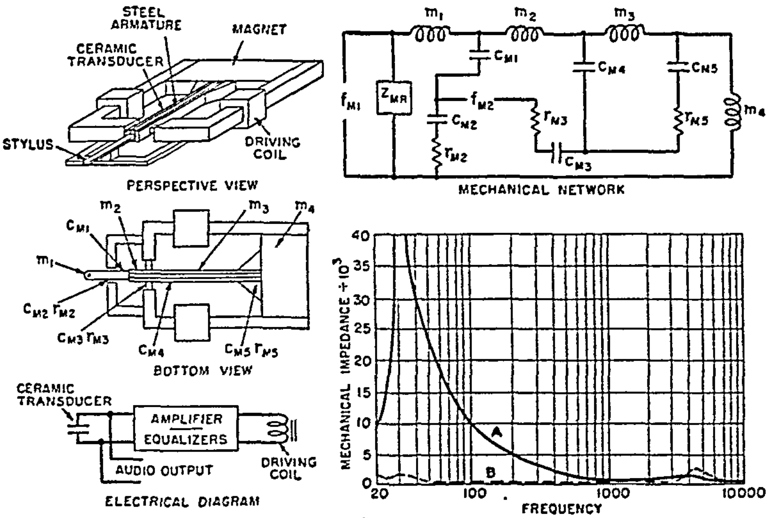


FIG. 13.11. Perspective and bottom views, mechanical network, electrical diagram and mechanical impedance frequency characteristics of a feedback pickup. In the mechanical network: z_{MR}, the mechanical impedance of the record. m_1, the mass of the stylus and stylus holder. C_{M1}, the compliance of the stylus arm. m_2, the mass of the front portion of the armature and ceramic transducer. r_{M2} and C_{M2}, the mechanical resistance and compliance of the damping block under the stylus arm. r_{M3} and C_{M3}, the mechanical resistance and compliance of the damping blocks on the ceramic transducer and magnetic armature. m_3, the mass of the rear portion of the ceramic transducer and magnetic armature. C_{M4}, the compliance of the ceramic transducer and magnetic armature. m_4, the mass of the pickup and tone arm. r_{M5} and C_{M5}, the mechanical resistance and compliance of the support for the ceramic transducer and armature. f_{M1} and f_{M2}, the forces generated by the velocity generator and the magnetic driving system. In the graph: A, mechanical impedance characteristic without feedback. B, mechanical impedance characteristic with feedback.

13.12. Free-Field Zone-Type Sound Reducer [10]

Existing conventional systems for the control and absorption of sound are of the passive type. Recently, active systems, in the form of electronic elements, have been developed for the control of sound and

[10] Olson and May, *Jour. Acous. Soc. Amer.*, Vol. 25, No. 6, p. 1130, 1953.

depicted in Fig. 13.12, as small as possible. Under these conditions the operation of the system is a sound pressure reducer. The amount of sound pressure reduction is a function of the distance between the microphone and loudspeaker, the wavelength of the sound wave, the phase relation in the electronic system and the distance from the microphone-loudspeaker combination. Typical experimental sound reduction frequency characteristics for various distances from the reducer are shown in Fig. 13.13. These characteristics show that the electronic sound re-

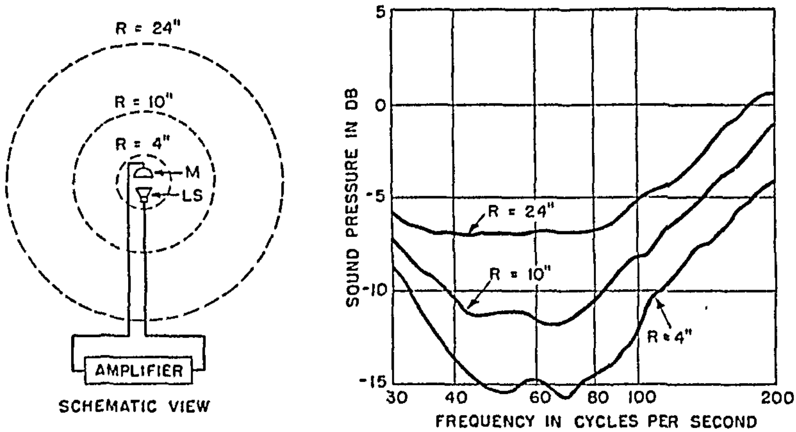


FIG. 13.13. Schematic view of an electronic sound reducer and sound pressure reduction frequency characteristics for distances of 4, 10 and 24 inches from the microphone-loudspeaker combination.

ducer may be used to reduce undesired sounds over a zone of operation.

One application for the electronic noise reducer is in the form of a noise reducer in airplanes and automobiles where the noise level is relatively high in the low-frequency range. With the practical use of conventional sound-absorbing materials in automobiles and airplanes the reduction in noise level in this frequency region is relatively small. For these applications the noise reducer may be installed on the back of the seat. There are also many other applications for the zone-type noise reducer, which include the reduction in noise from machines, heating apparatus, cooling and ventilating ducts and so forth.

13.13. Electronic Vibration Reducer [11]

Reduction in the transmission of sound through structures of solid materials is usually accomplished by the addition of mass or by a compliant isolating system. The latter means is usually preferred because the addition of mass is costly and, for most applications, impractical. The fundamental idea is to insert an element which has a low mechanical

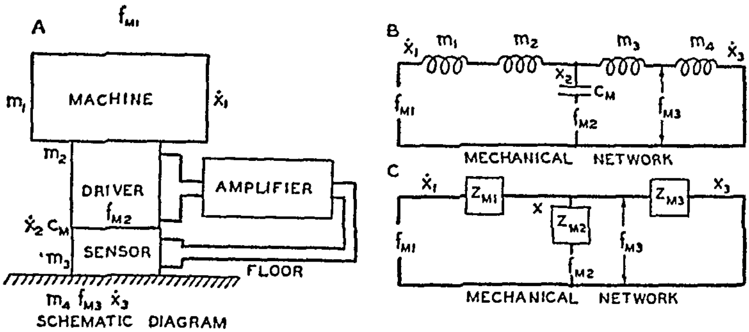


FIG. 13.14. A. A schematic diagram of a vibration reducer. B and C. Mechanical networks of a vibration reducer. In the mechanical network of B: m_1, the mass of the machine. m_2, the mass of the upper part of the driver. m_3, the mass of the lower part of the driver and sensor. m_4, the mass of the floor. C_M, the compliance of the driver and sensor. f_{M1}, the force developed by the machine. f_{M2}, the force developed by the driver. f_{M3}, the force developed at the floor. $\dot{x}_1$, the velocity of the machine. $\dot{x}_2$, the velocity developed by the driver. $\dot{x}_3$, the velocity developed in the floor. In the mechanical network of C: The mechanical impedance, z_{M1}, replaces the masses m_1 and m_2. The mechanical impedance, z_{M2}, replaces the compliance C_M. The mechanical impedance z_{M3} replaces the masses m_3 and m_4.

impedance and thereby provide a shunt for the vibrations. An electronic system may be used to provide the low mechanical impedance and thereby control and isolate the vibrations. In most problems involving the control of vibrations the amplitudes are relatively small and the mechanical impedance relatively large. Under these conditions piezoelectric transducers may be used. For example, an electronic vibration reducer may consist of a piezoelectric driver and sensor with a suitable amplifier. A system consisting of a piezoelectric driving system and piezoelectric sensor is shown in Fig. 13.14*A*. The mechanical network of the vibrating system is also shown in Fig. 13.14*B*. The mechanical network of Fig. 13.14*B* can be reduced to the mechanical

[11] Olson, H. F., *Jour. Acous. Soc. Amer.*, Vol. 25, No. 6, p. 1130, 1953.

network of Fig. 13.14*C*. The velocity $\dot{x}_3$ in the support may be expressed as follows:

$$\dot{x}_3 = \frac{f_{M1}z_{M2} - f_{M2}z_{M1}}{z_{M1}z_{M2} + z_{M1}z_{M3} + z_{M2}z_{M3}} \tag{13.22}$$

where z_{M1}, z_{M2} and z_{M3} are the mechanical impedances of the mechanical network of Fig. 13.14*C*.

From a consideration of equation 13.22, it will be seen that the magnitude of the velocity $\dot{x}_3$ can be reduced by the application of the force f_{M2} in the proper magnitude and phase with respect to f_{M1}. For example, $\dot{x}_3 = 0$ when

$$f_{M1}z_{M2} = f_{M2}z_{M1} \tag{13.23}$$

Under these conditions f_{M3} is also zero. That is to say, no vibrations are produced in the support. The machine is perfectly isolated from the support.

Another problem is to reduce the vibration of the machine without regard to the vibration transmitted to the support. The velocity $\dot{x}_1$ of the machine may be expressed as follows:

$$\dot{x}_1 = \frac{f_{M1}(z_{M2} + z_{M3}) - f_{M2}z_{M3}}{z_{M1}z_{M2} + z_{M1}z_{M3} + z_{M2}z_{M3}} \tag{13.24}$$

From a consideration of equation 13.24, it will be seen that the magnitude of the velocity $\dot{x}_1$ can be reduced by the application of the force f_{M2} in the proper magnitude and phase with respect to f_{M1}. For example, the velocity of the machine $\dot{x}_1$ will be zero if

$$f_{M1}(z_{M2} + z_{M3}) = f_{M2}z_{M3} \tag{13.25}$$

If $\dot{x}_1 = 0$, there will be no motion of the machine. However, under the conditions of feedback there will be a larger velocity $\dot{x}_3$ than in the absence of feedback.

There are many applications for an electronic vibration reducer which decreases the coupling between an offending vibration producer and a terminal location in which the vibrations are undesirable. Most applications for an electronic vibration reducer will involve the isolation of the vibrations produced by the machine from the foundation of the machine.

14.2. Definitions

A few of the terms used in dynamical mobility analogies will be defined in this section.

The mechanical terms *force, dyne, instantaneous force, effective force, maximum force, peak force, centimeter per second, instantaneous velocity, velocity, effective velocity, maximum velocity, peak velocity* and *mechanical rectilineal system*, which apply to mechanical mobility systems, have been defined in Chapters I and II and will not be repeated here.

Mechanical Rectilineal Mobility. The complex quotient of the alternating linear velocity applied to the system by the alternating linear force produced in the direction of the velocity at its point of application. The unit is the mechanical mho.

Mechanical Rectilineal Responsivity. The real part of the mechanical rectilineal mobility. This is the part responsible for the dissipation of energy. The unit is the mechanical mho.

Mechanical Rectilineal Excitability. The imaginary part of the mechanical rectilineal mobility. The unit is the mechanical mho.

Compliance in a mechanical rectilineal mobility system is that coefficient which, when multiplied by 2π times the frequency, gives the positive imaginary part of the mechanical rectilineal mobility. The unit is the centimeter per dyne.

Mass in the mechanical rectilineal mobility system is the coefficient which, when multiplied by 2π times the frequency, is the reciprocal of the negative imaginary part of the mechanical rectilineal mobility. The unit is the gram.

Mechanical Mobility Mho. A mechanical rectilineal responsivity, mechanical rectilineal excitability or mechanical rectilineal mobility is said to have a magnitude of one mechanical mho when a velocity of one centimeter per second produces a force of one dyne.

14.3. Mechanical Rectilineal Mobility

Mechanical rectilineal mobility is the inverse of mechanical rectilineal impedance. Mechanical rectilineal mobility z_I, in mechanical mhos, is defined as the complex ratio of linear velocity to linear force as follows:

$$z_I = \frac{v}{f_M} \qquad 14.1$$

where v = velocity, in centimeters per second, and
f_M = force, in dynes.

It will be evident that a mechanical element in the mechanical mobility sense is analogous to the electrical element if velocity difference across the mechanical element is analogous to the voltage difference across the electrical element and if the force through the mechanical element is analogous to the electrical current through the electrical element.

Mechanical rectilineal mobility z_I, in mechanical mhos, is a complex quantity and may be written as follows:

$$z_I = r_I + jx_I \qquad 14.2$$

where r_I = responsivity, in mechanical mhos, and
x_I = excitability, in mechanical mhos.

14.4. Responsivity (Mobility Resistance)

In the mechanical rectilineal mobility system mechanical rectilineal responsivity (mobility resistance) r_I, in mechanical mhos, is defined as

$$r_I = \frac{v}{f_M} = \frac{1}{r_M} \qquad 14.3$$

where v = velocity, in centimeters per second,
f_M = force, in dynes, and
r_M = mechanical impedance, in mechanical ohms.

14.5. Mass (Mobility Capacitance)

In the mechanical rectilineal mobility system the mass (mobility capacitance) m_I, in grams, is analogous to electrical capacitance C_E.

The mechanical rectilineal excitability x_I of a mass (mobility capacitance), in mechanical mhos, is defined as

$$x_I = \frac{1}{\omega m_I} \qquad 14.4$$

where $\omega = 2\pi f$, and
f = frequency, in cycles per second.

Equation 14.4 shows that the mass (mobility capacitance) m_I in the mechanical rectilineal mobility system is analogous to electrical capacitance C_E in the electrical system.

Mass (mobility capacitance) m_I in the mechanical rectilineal mobility system may also be defined as follows:

$$f_M = m_I \frac{dv}{dt} \qquad 14.5$$

$$v = \frac{1}{m_I} \int f_M dt \qquad 14.6$$

In the electrical system electrical capacitance C_E may be defined as follows:

$$i = C_E \frac{de}{dt} \qquad 14.7$$

where i = electrical current, in abamperes,
C_E = electrical capacitance, in abfarads,
e = electromotive force, in abvolts, and
t = time, in seconds.

$$e = \frac{1}{C_E} \int idt \qquad 14.8$$

where i = current, in abamperes.

It will be seen that equations 14.5 and 14.6 in the mechanical rectilineal mobility system are analogous to equations 14.7 and 14.8 in the electrical system.

14.6. Compliance (Mobility Inertia)

In the mechanical rectilineal mobility system the compliance (mobility inertia) C_I, in centimeters per dyne, is analogous to electrical inductance L.

The mechanical rectilineal excitability x_I of a compliance (mobility inertia), in mechanical mhos, is defined as

$$x_I = \omega C_I \qquad 14.9$$

where $\omega = 2\pi f$, and
f = frequency, in cycles per second.

Equation 14.9 shows that compliance (mobility inertia) C_I, in centimeters per dyne, is analogous to inductance.

Compliance (mobility inertia) C_I in the mechanical rectilineal mobility system may also be defined as

$$v = C_I \frac{df_M}{dt} \tag{14.10}$$

In the electrical system inductance may be defined as

$$e = L \frac{di}{dt} \tag{14.11}$$

where L = inductance, in abhenries.

It will be seen that equation 14.10 in the mechanical rectilineal mobility system is analogous to equation 14.11 in the electrical system.

14.7. Representation of Electrical and Mechanical Rectilineal Mobility Elements

Electrical elements have been defined in Chapter II. Elements in the mechanical rectilineal mobility system have been described in the preceding sections of this chapter.

Fig. 14.1 illustrates schematically the mechanical elements and the analogous elements in the electrical and mechanical rectilineal mobility systems.

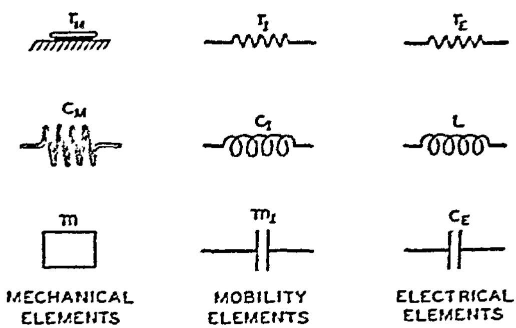


FIG. 14.1. Graphical representation of the three basic elements in mechanical rectilineal, mobility and electrical systems.

r_m = mechanical rectilineal resistance — r_I = responsivity — r_E = electrical resistance

C_M = compliance — C_I = mobility inertia — L = inductance

m = mass — m_I = mobility capacitance — C_E = electrical capacitance

Mechanical rectilineal resistance r_M in the mechanical rectilineal system is represented as sliding or viscous friction. Mechanical rectilineal responsivity (mobility resistance) r_I in the mechanical rectilineal mobility system is the reciprocal of mechanical rectilineal resistance r_M and is analogous to electrical resistance r_E.

Compliance C_M in the mechanical rectilineal system is represented as a spring. Compliance (mobility inertia) C_I in the mechanical rectilineal mobility system is analogous to inductance L in the electrical system.

TABLE 14.1

Electrical			Mechanical Rectilineal Mobility		
Quantity	Unit	Symbol	Quantity	Unit	Symbol
Electromotive Force	Volts $\times 10^{-8}$	e	Velocity	Centimeters per Second	$\dot{x}$ or v
Charge or Quantity	Coulombs $\times 10^{-1}$	q	Impulse or Momentum	Gram Centimeter per Second	Q
Current	Amperes $\times 10^{-1}$	i	Force	Dynes	f_M
Electrical Impedance	Ohms $\times 10^9$	z_E	Mechanical Mobility	Mechanical Mhos	z_I
Electrical Resistance	Ohms $\times 10^9$	r_E	Responsivity	Mechanical Mhos	r_I
Electrical Reactance	Ohms $\times 10^9$	x_E	Excitability	Mechanical Mhos	x_I
Inductance	Henries $\times 10^9$	L	Compliance or Mobility Inertia	Centimeters per Dyne	C_I
Electrical Capacitance	Farads $\times 10^9$	C_E	Mass or Mobility Capacitance	Grams	m_I
Power	Ergs per Second	P_E	Power	Ergs per Second	P_I

Mass m in the mechanical rectilineal system is represented as a mass or weight. Mass (mobility capacitance) m_I in the mechanical rectilineal mobility system is analogous to electrical capacitance C_E in the electrical system.

The quantities in the electrical and mechanical rectilineal mobility systems are shown in Table 14.1. The units and the analogous elements and symbols in the two systems are shown in Table 14.1.

14.8. Mechanical Vibrating Systems Consisting of a Mass, Compliance and Mechanical Resistance

The vibrating system [3] of one degree of freedom consisting of a mass, compliance and mechanical resistance has been considered from the standpoint of the classical mechanical impedance analogy in Chapter III. It is the purpose of this section to consider the same and other mechanical vibrating systems from the standpoint of the mechanical mobility analogy.[4]

Six mechanical vibrating systems consisting of a mass, compliance and mechanical resistance in different arrangements will be considered in this section. Two of the mechanical vibrating systems—namely, those shown in Figs. 14.2 and 14.4—have been considered in Chapters III and IV. Four of the mechanical vibrating systems—namely, those shown in Figs. 14.3, 14.5, 14.6 and 14.7—have not been considered in the preceding sections concerned with classical mechanical impedance analogy. Therefore, the classical mechanical network of these vibrating systems will be included in this section.

The first mechanical system consisting of a mass, compliance and mechanical resistance is shown in Fig. 14.2*A*. The mechanical vibrating system may be rearranged to form the equivalent as shown in Fig. 14.2*B*. From the mechanical vibrating system of Fig. 14.2*B* it is a relatively simple matter to develop the mobility analogy of Fig. 14.2*C*.

---

[3] The preceding sections have been concerned with fundamental considerations. Therefore, the modifier rectilineal has been employed for the sake of accuracy. Since the remainder of this chapter will be concerned with applications of the mechanical rectilineal mobility, the modifier rectilineal will be dropped.

[4] In view of the fact that this chapter is concerned with mechanical systems, the modifier mechanical in relation to the mechanical mobility analogy is also superfluous and need not be used.

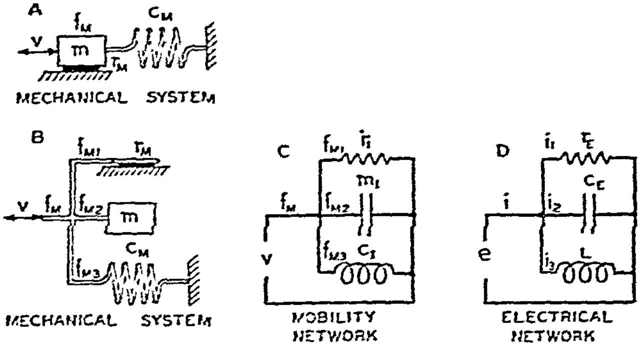


FIG. 14.2. A mechanical vibrating system consisting of a mass, compliance and mechanical resistance. A. Mechanical system. B. Mechanical system equivalent to the mechanical system of A. C. Mobility network of the mechanical system. D. Electrical network analog of the mobility system.

The sum of the forces through the three branches of the mobility network[5] of Fig. 14.2*C* is

$$f_M = f_{M1} + f_{M2} + f_{M3} \qquad 14.12$$

where

$$f_{M1} = \frac{v}{r_I} \qquad 14.13$$

$$f_{M2} = m_I \frac{dv}{dt} \qquad 14.14$$

$$f_{M3} = \frac{1}{C_I} \int v dt \qquad 14.15$$

[5] In establishing analogies between electrical and mechanical systems the elements in the electrical network have been labeled r_E, L and C_E. However, in using analogies in actual practice the conventional procedure is to label the elements in the analogous electrical network with r_M, m and C_M for the classical mechanical rectilineal system and with r_I, C_I and m_I for the mobility mechanical rectilineal system. This procedure will be followed in this chapter in labeling the elements of the analogous electrical network. It is literally accurate to label the network with the caption "Analogous electrical network of the mechanical rectilineal system" (or, of the mobility mechanical rectilineal system). For the sake of brevity, these networks will be labeled "mechanical network" and "mobility network." Where there is only one path, "circuit" will be used instead of "network."

From the sum of equations 14.13, 14.14 and 14.15 the differential equation of the mobility network of Fig. 14.2*C* is

$$f_M = m_I \frac{dv}{dt} + \frac{v}{r_I} + \frac{1}{C_I} \int v dt \qquad 14.16$$

The sum of the electrical currents of the electrical network of Fig. 14.2*D* is

$$i = i_1 + i_2 + i_3 \qquad 14.17$$

where

$$i_1 = \frac{e}{r_E} \qquad 14.18$$

$$i_2 = C_E \frac{de}{dt} \qquad 14.19$$

$$i_3 = \frac{1}{L} \int e dt \qquad 14.20$$

From the sum of equations 14.18, 14.19 and 14.20 the differential equation of the electrical network of Fig. 14.2*D* is

$$i = C_E \frac{de}{dt} + \frac{e}{r_E} + \frac{1}{L} \int e dt \qquad 14.21$$

Comparing the variables and coefficients of the mobility and electrical networks in the differential equations 14.16 and 14.21 establishes the analogous variables and quantities in the two systems as given in Table 14.1.

The classical mechanical impedance analogy of the mechanical system of Fig. 14.2 has been considered in Chapter III and will not be repeated here.

The second mechanical system consisting of a mass, compliance and mechanical resistance is shown in Fig. 14.3*A*. New labels for establishing the mobility analogy are shown in Fig. 14.3*B*. The mechanical mobility circuit of the mechanical vibrating system is shown in Fig. 14.3*C*. In the mobility analogy the force f_M is the same through all the elements of the mobility circuit of Fig. 14.3*C*. The velocity drop through the three elements of the mobility circuit of Fig. 14.3*C* is

$$v = v_1 + v_2 + v_3 \qquad 14.22$$

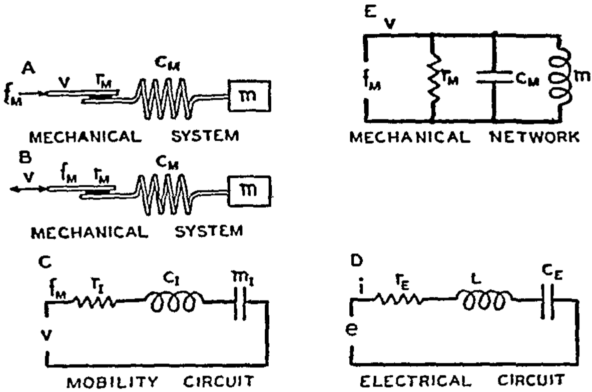


FIG. 14.3. A mechanical vibrating system consisting of a mass, compliance and mechanical resistance. A. Mechanical system. B. Mechanical system equivalent to the mechanical system of A. C. Mobility circuit of the mechanical system. D. Electrical circuit analog of the mobility system. E. Mechanical network in the classical impedance analogy system.

where
$$v_1 = r_I f_M \tag{14.23}$$
$$v_2 = C_I \frac{df_M}{dt} \tag{14.24}$$
$$v_3 = \frac{1}{m_1} \int f_M dt \tag{14.25}$$

From the sum of equations 14.23, 14.24 and 14.25 the differential of the mobility circuit is

$$v = \frac{1}{m_1} \int f_M dt + r_I f_M + C_i \frac{df_M}{dt} \tag{14.26}$$

The current i is the same through all the elements in the electrical circuit of Fig. 14.3D. The voltage drop across through the three elements of Fig. 14.3D is

$$e = e_1 + e_2 + e_3 \tag{14.27}$$

where
$$e_1 = r_E i \tag{14.28}$$
$$e_2 = L \frac{di}{dt} \tag{14.29}$$
$$e_3 = \frac{1}{C_E} \int i dt \tag{14.30}$$

From the sum of equations 14.30, 14.31 and 14.32 the differential of the electrical circuit of Fig. 14.3C is

$$e = \frac{1}{C_E}\int idt + r_E i + L\frac{di}{dt} \qquad 14.31$$

The solution of the differential equation 14.31 is

$$i = \frac{e}{r_E + j\omega L + \dfrac{1}{j\omega C_E}} = \frac{e}{z_E} \qquad 14.32$$

The electrical impedance z_E is

$$z_E = r_E + j\omega L + \frac{1}{j\omega C_E} \qquad 14.33$$

The solution of the differential equation 14.26 is

$$f_M = \frac{v}{r_I + j\omega C_I + \dfrac{1}{j\omega m_I}} = \frac{v}{z_I} \qquad 14.34$$

The mechanical mobility z_I is

$$z_I = r_I + j\omega C_I + \frac{1}{j\omega m_I} \qquad 14.35$$

The electrical quantities in equation 14.33 and the mobility quantities in equation 14.35 are defined in Table 14.1.

Comparing the variables and coefficients of the mobility and electrical circuits in the differential equations 14.26 and 14.31 establishes the analogous variables and quantities in the two systems as given in Table 14.1.

The solution of the differential equations yields the equations for the mechanical mobility and electrical impedance. These are given in equations 14.33 and 14.35. A comparison of the variables and quantities in the expressions for the mechanical mobility and electrical impedance provides another example of the analogy of mechanical mobility systems and electrical systems.

The classical mechanical impedance network of the mechanical system of Fig. 14.3A is shown in Fig. 14.3E. The mechanical network of Fig.

14.3*E* can be established and solved by employing the procedures outlined in the preceding chapters of this book.

The third mechanical system consisting of a mass, compliance and mechanical resistance is shown in Fig. 14.4*A*. For convenience in

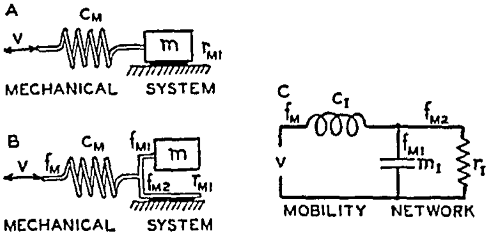


FIG. 14.4. A mechanical vibrating system consisting of a mass, compliance and mechanical resistance. A. Mechanical system. B. Mechanical system equivalent to the mechanical system of A. C. Mobility network of the mechanical system.

establishing the mobility networks the mechanical system may be rearranged to the equivalent mechanical system of Fig. 14.4*B*; the mechanical mobility network is shown in Fig. 14.4*C*. By employing the expressions for the quantities and elements which have been established, the performance of the system may be obtained from the mobility network. The classical impedance analogy of the system of Fig. 14.4 has been considered in Chapter IV and will not be repeated here.

The fourth mechanical system consisting of a mass, compliance and mechanical resistance is shown in Fig. 14.5*A*. The classical mechanical

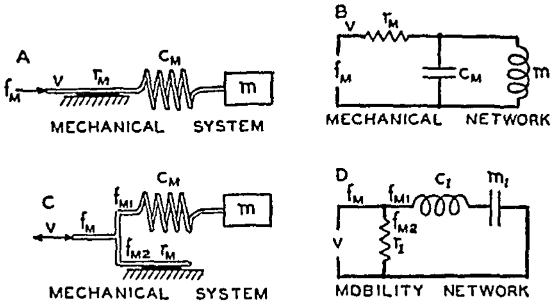


FIG. 14.5. A mechanical vibrating system consisting of a mass, compliance and mechanical resistance. A. Mechanical system. B. Mechanical network in the classical impedance analogy system. C. Mechanical system equivalent to the mechanical system of A. D. Mobility network of the mechanical system.

network of the mechanical vibrating system is shown in Fig. 14.5*B*. The establishment and solution of the mechanical network can be determined by procedures as outlined in the preceding chapters of this book. As an aid in establishing the mobility, the mechanical vibrating system is redrawn as shown in Fig. 14.5*C*; the mobility network of the mechanical vibrating system is shown in Fig. 14.5*D*.

The fifth mechanical system consisting of a mass, compliance and mechanical resistance is shown in Fig. 14.6*A*. The classical mechanical

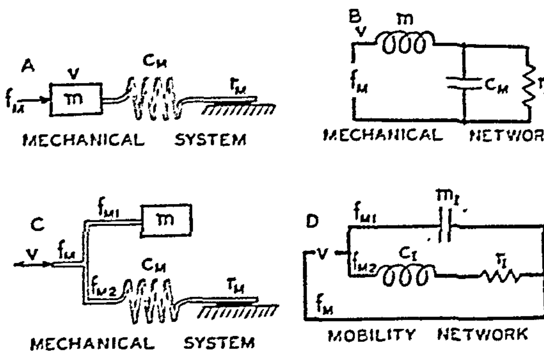


FIG. 14.6. A mechanical vibrating system consisting of a mass, compliance and mechanical resistance. A. Mechanical system. B. Mechanical network in the classical impedance analogy system. C. Mechanical system equivalent to the mechanical system of A. D. Mobility network of the mechanical system.

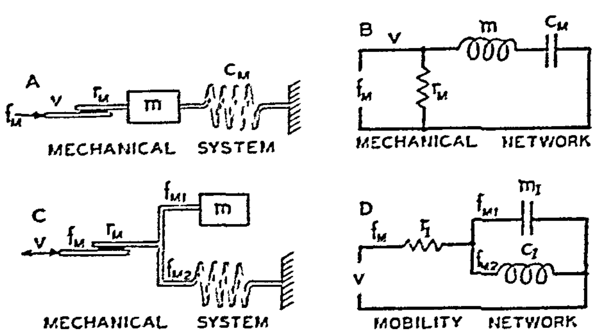


FIG. 14.7. A mechanical vibrating system consisting of a mass, compliance and mechanical resistance. A. Mechanical system. B. Mechanical network in the classical impedance analogy system. C. Mechanical system equivalent to the mechanical system of A. D. Mobility network of the mechanical system.

network of the mechanical vibrating system is shown in Fig. 14.6*B*. The same mechanical system, redrawn as an aid in establishing the mobility network, is shown in Fig. 14.6*C*; the mobility network of the mechanical vibrating system is shown in Fig. 14.6*D*.

The sixth mechanical system consisting of a mass, compliance and mechanical resistance is shown in Fig. 14.7*A*. The classical mechanical network of the mechanical vibrating system is shown in Fig. 14.7*B*. The mobility network of the mechanical vibrating system of Fig. 14.7*C*, which is equivalent to mechanical vibrating system of Fig. 14.7*A*, is shown in Fig. 14.7*D*.

14.9. Mechanical Vibrating System of Three Degrees of Freedom

A mechanical vibrating system of three degrees of freedom has been considered from the standpoint of the classical impedance analogy in Sect. 4.13. The mechanical vibrating system of three degrees of freedom is shown in Fig. 14.8*A*. The mobility network of the mechanical vibrating system of Fig. 14.8*A* is shown in Fig. 14.8*B*; the electrical network analogous to the mobility network is shown in Fig. 14.8*C*.

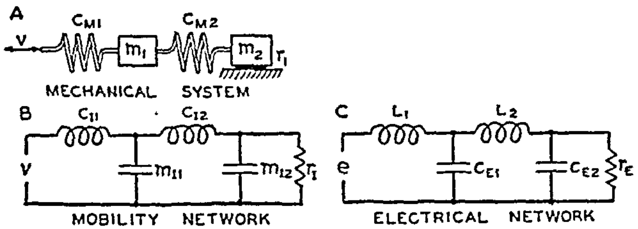


FIG. 14.8. A mechanical vibrating system consisting of two masses, two compliances and a mechanical resistance. A. Mechanical system. B. Mobility network of the mechanical system. C. Electrical network analog of the mobility system.

14.10. Electrical, Mechanical and Mobility Transformers

A transformer is a transducer used for transforming between two impedances without appreciable reflection loss.

Electrical and mechanical transformers have been considered from the standpoint of the classical impedance analogy in Sect. 5.22. The lever shown in Fig. 14.9*A* is a mechanical transformer; the mobility transformer is shown in Fig. 14.9*B*. The electrical transformer analogous

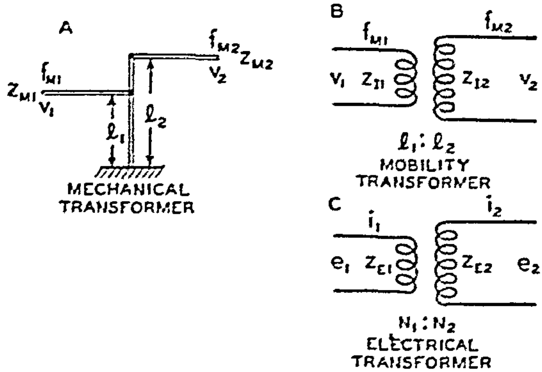


FIG. 14.9. Mechanical, mobility and electrical transformers. A. Mechanical transformer. B. Mobility transformer of the mechanical system. C. Electrical transformer analog of the mobility transformer.

to the mechanical transformer in the mobility complex is shown in Fig. 14.9*C*. The ratios of the lengths of the levers and number of turns in the mechanical and electrical systems are as follows:

$$l_1 : l_2 = N_1 : N_2 \qquad 14.36$$

where l_1 and l_2 = the lengths of the levers shown in Fig. 14.9*A*, and
N_1 and N_2 = turns on the primary and secondary of the electrical transformer in Fig. 14.9*C*.

14.11. Wave Filters

Wave filters have been considered from the classical impedance analogy in Chapter VI. It is the purpose of this section to describe the performance of low and high pass wave filters from the standpoint of the mobility analogy.

A low pass mechanical wave filter is shown in Fig. 14.10*A*. The mobility and electrical wave filters which are analogies of the mechanical wave filters are shown in Fig. 14.10*B* and *C*.

The mechanical mobility of the series arm of Fig. 14.10*B* is given by

$$z_{I1} = j\omega C_I \qquad 14.37$$

The electrical impedance of the series arm of Fig. 14.10*C* is given by

$$z_{E1} = j\omega L \qquad 14.38$$

The mechanical mobility of the center shunt arm of Fig. 14.10*B* is given by

$$z_{I2} = \frac{1}{j\omega m_I} \qquad 14.39$$

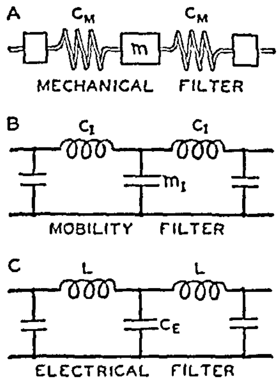


FIG. 14.10. Mechanical, mobility and electrical low pass wave filters. A. Mechanical low pass filter. B. Mobility low pass filter of the mechanical system. C. Electrical low pass filter analog of the mobility low pass filter.

The electrical impedance of the center shunt arm of Fig. 14.10*C* is given by

$$z_{E2} = \frac{1}{j\omega C_E} \qquad 14.40$$

The limiting frequencies of transmission are given by

$$\frac{z_1}{z_2} = 0 \quad \text{and} \quad \frac{z_1}{z_2} = -4 \qquad 14.41$$

From the constants of the systems

$$\frac{z_{I1}}{z_{I2}} = C_I m_I \omega_C^2 = 0 \qquad \text{when } \omega_C = 0 \qquad 14.42$$

$$\frac{z_{E1}}{z_{E2}} = LC_E \omega_C^2 = 0 \qquad \text{when } \omega_C = 0 \qquad 14.43$$

$$\frac{z_{I1}}{z_{I2}} = -C_I m_I \omega_C^2 = -4 \quad \text{when } \omega_C = \frac{2}{\sqrt{C_I m_I}} \qquad 14.44$$

$$\frac{z_{E1}}{z_{E2}} = -LC_E\omega_C^2 = -4 \quad \text{when } \omega_C = \frac{2}{\sqrt{LC_E}} \qquad 14.45$$

Equations 14.42 to 14.45 inclusive show that the systems of Fig. 14.10 are low pass filters transmitting currents of all frequencies lying between 0 and the upper cutoff frequency f_C where $f_C = \dfrac{\omega_C}{2\pi}$.

A high pass mechanical wave filter is shown in Fig. 14.11*A*. The mobility and electrical wave filters which are analogies of the mechanical filter are shown in Fig. 14.11*B* and *C*.

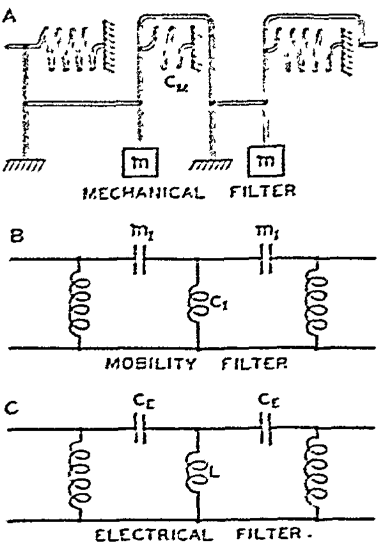


FIG. 14.11. Mechanical, mobility and electrical high pass wave filters. A. Mechanical high pass filter. B. Mobility high pass filter of the mechanical system. C. Electrical high pass filter analog of the mobility high pass filter.

The mechanical mobility of the series arm of Fig. 14.11*B* is given by

$$z_{I1} = \frac{1}{j\omega m_I} \qquad 14.46$$

The electrical impedance of the series arm of Fig. 14.11*C* is given by

$$z_{E1} = \frac{1}{j\omega C_E} \qquad 14.47$$

The mechanical mobility of the shunt arm of Fig. 14.11*B* is given by

$$z_{I2} = j\omega C_I \qquad 14.48$$

The electrical impedance of the shunt arm of Fig. 14.11*C* is given by

$$z_{E2} = j\omega L \qquad 14.49$$

The limiting frequencies of transmission are given by

$$\frac{z_1}{z_2} = 0 \quad \text{and} \quad \frac{z_1}{z_2} = -4$$

From the constants of the system

$$\frac{z_{I1}}{z_{I2}} = -\frac{1}{C_I m_I \omega_C^2} = 0 \qquad \text{when } \omega_C = \infty \qquad 14.50$$

$$\frac{z_{E1}}{z_{E2}} = -\frac{1}{LC_E\omega_C^2} = 0 \qquad \text{when } \omega_C = \infty \tag{14.51}$$

$$\frac{z_{I1}}{z_{I2}} = -\frac{1}{C_I m_I \omega_C^2} = -4 \quad \text{when } \omega_C = \frac{1}{2\sqrt{C_I m_I}} \tag{14.52}$$

$$\frac{z_{E1}}{z_{E2}} = -\frac{1}{LC_E\omega_C^2} = -4 \quad \text{when } \omega_C = \frac{1}{2\sqrt{LC_E}} \tag{14.53}$$

Equations 14.50 to 14.53 inclusive show that the systems of Fig. 14.11 are high pass filters transmitting currents and mechanical forces of all frequencies lying between the lower cutoff frequency f_C where $f_C = \frac{\omega_C}{2\pi}$ and infinity.

14.12. Driving Systems

The subject of driving systems from the standpoint of the classical impedance analogies has been considered in Chapter VIII. These systems may be also considered from the standpoint of the mobility analogy. It is the purpose of this section to show that there are advantages for each analogy depending upon the type of transducer.

A. *Electrodynamic Driving System.*—A moving coil or dynamic driving system is a driving system in which the mechanical forces are developed by the interaction of currents in a conductor and the magnetic field in which it is located. The electrodynamic system is depicted in Fig. 14.12, where the electrical and mobility networks are also shown.

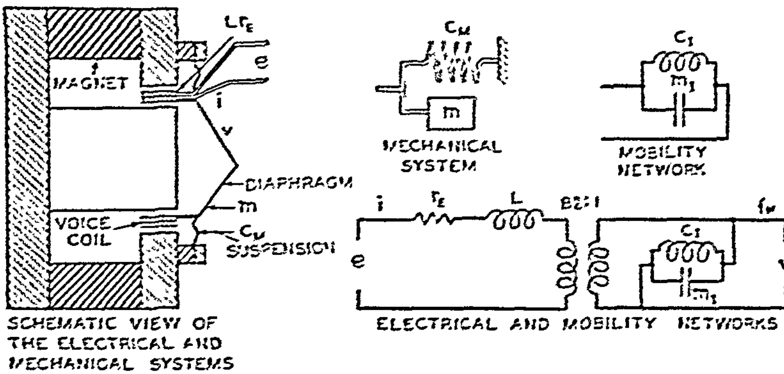
The electromotive force, in abvolts, on the electrical side of the system is the sum of the electromotive force drop due to the current in the electrical system plus the back electromotive force developed due to the motion of the system. Sect. 8.2. This may be expressed as follows:

$$e = (r_E + j\omega L)i + Bl\dot{x} \tag{14.54}$$

where e = electromotive force applied at the electrical input, in abvolts,
r_E = electrical resistance of the voice coil, in abohms,
L = inductance of the voice coil, in abhenries,
$\omega = 2\pi f$,
f = frequency, in cycles per second,
i = current in the voice coil, in abamperes,

B = flux density in the air gap, in gausses,
l = length of the voice coil conductor, in centimeters, and
$\dot{x} = v$ = velocity of the voice coil, in centimeters per second.

The force developed on the mechanical side is the difference between the force developed due to motion (employing the mobility analogy,



equation 14.1) and the force due to the current in the coil. This may be expressed as follows:

$$f_M = \frac{v}{z_I} - Bli \qquad 14.55$$

where f_M = force, in dynes,
v = velocity of the system, in centimeters per second, and
z_I = mechanical mobility, in mechanical mhos.

The mechanical mobility of the mechanical system is given by

$$\frac{1}{z_I} = j\omega m_I + \frac{1}{j\omega C_I} \qquad 14.56$$

where C_I = compliance or mobility inertia of the suspension, in centimeters per dyne, and

m_I = mass or mobility capacitance of the diaphragm, in grams.

In the complete electrical and mobility network there is an ideal transformer coupling the electrical and mechanical systems with a turns ratio of Bl:1.

From the complete network representation of the electrical and mechanical system shown in Fig. 14.12 it may be deduced that there is some advantage in the use of mobility analogy because electromotive force and current are analogous to velocity and force, respectively.

B. *Electrostatic Driving System.*—An electrostatic driving system is a driving system in which the mechanical forces are produced from elec-

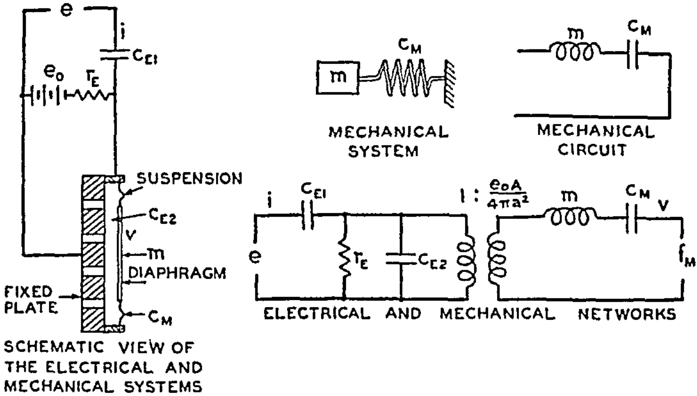


FIG. 14.13. Electrostatic driving system, the mechanical system, the mechanical circuit and the electrical and mechanical networks. In the electrical and mechanical networks: e, the signal electromotive force applied to the system. i, the current due to the applied electromotive force. C_{E1}, the electrical capacitance of the blocking capacitor. r_E, the electrical resistance of the polarizing resistor. e_0, the electromotive force of the polarizing battery. C_{E2}, the electrical capacitance of the electrostatic transducer. m, the mass of the diaphragm. C_M, the compliance of the suspension system. v, the velocity of the diaphragm. f_M, the force developed in the mechanical system.

trostatic reaction. The electrostatic system consisting of a fixed and movable plate and an electrical polarizing means is depicted in Fig. 14.13. The electrical and classical mechanical impedance networks are shown in Fig. 14.13.

The driving force upon the plate developed by the applied electromotive force is given by equation 8.58 as follows:

$$f_M = \frac{e_0 e A}{4\pi a^2} \qquad 14.57$$

where f_M = driving force, in dynes,
e_0 = polarizing electromotive force, in statvolts,
e = driving electromotive force, in statvolts,
A = area of the movable plate, in square centimeters, and
a = spacing between the movable and stationary plate, in centimeters.

The force developed on the mechanical side is the sum of the force developed by the applied voltage and the force due to motion. This may be expressed as follows:

$$f_M = \frac{e_0 e A}{4\pi a^2} + z_M \dot{x} \qquad 14.58$$

where f_M = force, in dynes,
z_M = mechanical impedance, in mechanical ohms, and
$\dot{x} = v$ = velocity of the plate, in centimeters per second.

The mechanical impedance of the vibrating system is given by

$$z_M = j\omega m + \frac{1}{j\omega C_M} \qquad 14.59$$

where z_M = mechanical impedance, in mechanical ohms,
m = mass of the diaphragm, in grams, and
C_M = compliance of the suspension, in centimeters per dyne.

The current generated by the movement of the plate is given by equation 8.67 as follows:

$$i = \frac{e_0 A}{4\pi a^2} \dot{x} \qquad 14.60$$

where i = current, in statamperes, and
$\dot{x}$ = velocity, in centimeters per second.

The complete equation for the current in the electrical circuit, neglecting the polarizing system, is given by

$$i = \frac{e_0 A}{4\pi a^2}\dot{x} + j\omega C_E e \tag{14.61}$$

where C_E = electrical capacitance, in statfarads.

In the complete electrical and mechanical network there is an ideal transformer with a turns ratio of $1{:}\frac{Ae_0}{4\pi a^2}$.

In the complete network representation of the electrical and mechanical system shown in Fig. 14.13 it may be deduced that there is some advantage in the use of the classical mechanical impedance analogy because electromotive force and current are analogous to force and velocity, respectively.

C. *Use of the Classical and Mobility Analogies.*—The examples in this section have illustrated that advantages in the use of the classical impedance and mobility analogies reside in the type of transducer. The use of the mobility analogy results in a direct analogy for dynamic, magnetic and magnetostriction transducers, while the use of the classical impedance analogy results in a direct analogy for electrostatic and crystal transducers.

14.13. Direct Radiator Loudspeaker

The direct radiator dynamic loudspeaker shown in Fig. 14.14*A* is almost universally used for radio, phonograph, television and other small-scale sound reproduction.

The electrical and mechanical systems of the complete loudspeaker are shown in Fig. 14.14*A*. The mechanical vibrating system consisting of the voice coil, cone, suspension and air load is shown in Fig. 14.14*B*.

The mass m_1 of the cone and voice coil, and the compliance C_M and mechanical resistance of the suspension system, can be obtained from measurements of the vibrating system.

The mechanical system of the air load—namely, the mechanical resistance r_{M2} and mass m_2 of the air load upon the front of the cone—is depicted in Figs. 14.15*A* and 14.15*D*. The mechanical network of the air load upon the front of the cone is shown in Fig. 14.15*B*. The constants of the mechanical resistance and mass of the air load upon

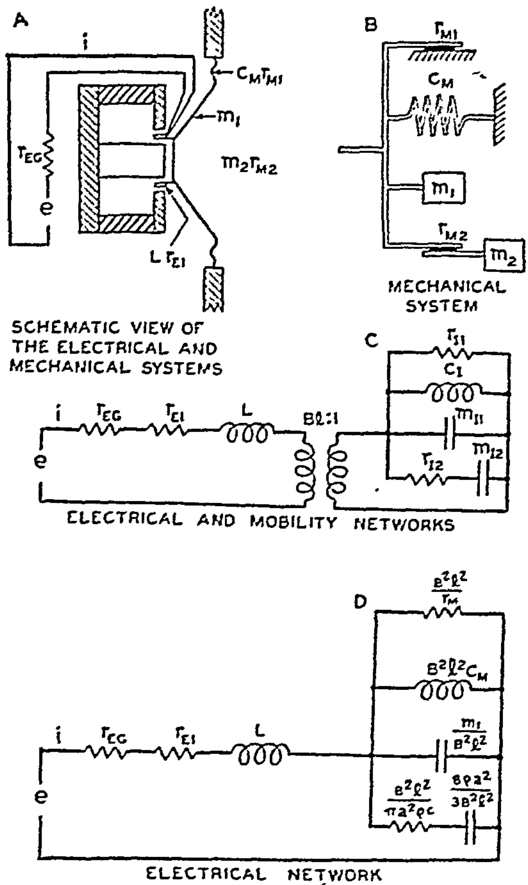


FIG. 14.14. Cross-sectional view, the mechanical system, the electrical and mobility networks and the electrical network of a direct radiator dynamic loudspeaker. In the electrical and mechanical networks: e, the electromotive force of the electrical generator. r_{EG}, the electrical resistance of the electrical generator. L, the inductance of the voice coil. r_{E1}, the electrical resistance of the voice coil. m_1, the mass of the cone. C_M and r_{M1}, the compliance and mechanical resistance of the suspension. m_2 and r_{M2}, the mass and mechanical resistance of the air load. m_I, the mobility capacitance of the cone. C_I and r_{I1}, the mobility inertia and responsivity of the suspension. m_{I2} and r_{I2}, the mobility capacitance and responsivity of the air load. B, the flux density in the air gap. l, the length of the voice coil conductor. a, the radius of the cone. ρ, density of the air.

the front of the cone are shown in the mechanical network of Fig. 14.15C. The mobility circuit of the air load upon the front of the cone is shown

in Fig. 14.15*E*. The constants of the responsivity and compliance are shown in the mobility circuit of Fig. 14.15*F*.

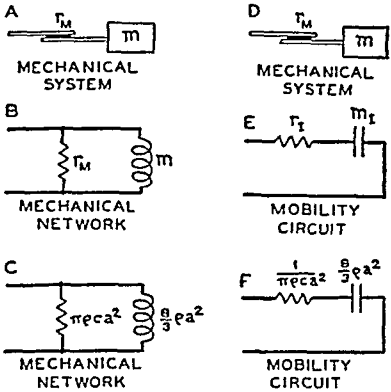


FIG. 14.15. Air load upon a loudspeaker cone. A. Mechanical system. m, the mass of the air load. r_M, the mechanical resistance of the air load. B. Mechanical network of the air load upon a loudspeaker cone. C. Mechanical network of the air load upon a loudspeaker cone. a, the radius of the cone. ρ, the density of air. c, the velocity of sound. D. Mechanical system same as A. E. Mobility circuit of the air load upon a loudspeaker cone. m_I, the mobility capacitance of the air load. r_I, the responsivity of the air load. F. Mobility circuit of the air load upon a loudspeaker cone. a, the radius of the cone. ρ, the density of air. c, the velocity of sound.

The electrical and mobility networks with the ideal transformer connecting the electrical and mobility sections are shown in Fig. 14.14*C*.

In Fig. 14.14*D* the ideal transformer has been eliminated, and the entire vibrating system reduced to an electrical network. The electrical impedance due to the mechanical system is given by equation 8.4 as follows:

$$z_{EM} = \frac{(Bl)^2}{z_M} \qquad 14.62$$

where z_{EM} = electrical impedance due to the mechanical system, in abohms,

z_M = mechanical impedance of the mechanical system, in mechanical ohms,

B = flux density in the air gap, in gausses, and

l = length of the voice coil conductor, in centimeters.

Since $\frac{1}{z_M} = z_I$, equation 14.62 may be written

$$z_{EM} = (Bl)^2 z_I \tag{14.63}$$

where z_I = mobility, in mechanical mhos.

By means of equation 14.63 it is possible to convert the combined electrical and mobility networks to the electrical network, as shown in Fig. 14.14*D*.

The process employing the mobility analysis of this section may be compared with the classical impedance analysis of Sect. 11.4.

CHAPTER XV

MAGNETIC ANALOGY

15.1. Introduction

Magnetic systems are important elements in all manner of transducers employed in sound, control, servo, generating, driving and feedback systems. These transducers include loudspeakers, microphones, phonographs, phonograph cutters and pickups, magnetic heads, film recording galvanometers, telephone receivers, meters, relays, motors, generators, solenoids and so forth. As in the case of mechanical and acoustical systems it is possible to establish analogies between electrical and magnetic systems. The use of analogies between electrical and magnetic systems and the reduction of a magnetic system to the analogous electrical network are powerful tools for use in the solution of problems in magnetic systems. It is the purpose of this chapter to develop the analogy between electrical and magnetic systems.

15.2. Definitions

The most important terms used in magnetic systems will be defined in this section as follows:

Magnetic Flux. The physical manifestation of a condition existing in a medium or material subjected to a magnetizing influence. The quantity is characterized by the fact that an electromotive force is induced in a conductor surrounding the flux during any time that the flux changes in magnitude. In the cgs system the unit of magnetic flux is the maxwell.

Magnetomotive Force in a magnetic circuit is the work required to carry a unit magnetic pole around the circuit against the magnetic field. In the cgs system the unit of magnetomotive force is the gilbert.

Reluctance. The property of the magnetic circuit to resist magnetization. Thus the amount of magnetic flux resulting from a given magnetomotive force acting on a magnetic circuit is determined by the

magnetic reluctance of the circuit. In the cgs system the unit is the gilbert per maxwell.

Maxwell. The cgs unit of magnetic flux. It is the flux produced by a magnetomotive force of one gilbert in a magnetic circuit of unit reluctance.

Line. A term commonly used interchangeably for a maxwell.

Gilbert. The cgs unit of magnetomotive force. It is the magnetomotive force required to produce one maxwell of magnetic flux in a magnetic circuit of unit reluctance.

Oersted. The unit of field strength in the cgs system. It is the magnetomotive force equivalent to one gilbert per centimeter of length. One oersted is the magnetic field strength at the center of a plane circular coil of one turn and a radius of one centimeter carrying a current of $\pi/2$ abamperes.

Gauss. The unit of flux density. One gauss equals one maxwell per square centimeter.

Flux. The term applied to the physical manifestation of the presence of magnetic induction.

Flux Density. The number of lines or maxwells per unit area in a section normal to the direction of the flux. In the cgs system the unit is the gauss.

Ampere Turn. The unit of magnetomotive force. It is a product of the number of turns on a coil and the amperes passing through the turns.

Magnetizing Force. The magnetomotive force per unit length at any given point in a magnetic circuit. In the cgs system the unit of magnetizing force is the oersted.

Leakage. That portion of the magnetic field that is not useful.

Leakage Coefficient. The ratio of the total flux produced to the useful flux.

Induction, Intrinsic (Ferric Induction). That portion of the induction in excess of the induction in a vacuum for the same magnetizing force.

Induction, Magnetic. The magnetic flux per unit area of a section normal to the direction of flux resulting when a substance is subjected to a magnetic field. This is also known as magnetic flux density. In the cgs system the unit of magnetic flux density is the gauss.

Coercive Force. The magnetomotive force which must be applied to a magnetic material in a direction opposite to the residual induction to reduce the latter to zero.

Demagnetization. The reduction of magnetization. It may be either partial or complete.

Demagnetization Curve. That portion of the normal hysteresis loop in the second quadrant showing the induction in a magnetic material as related to the magnetizing force applied in a direction opposite to the residual induction. See Fig. 15.9.

Permeability. The ratio of the magnetic induction in a given medium to the induction which would be produced in a vacuum by the action of the same magnetizing force. The magnetic circuit in the medium must be a closed circuit with no leakage.

15.3. Simple Magnetic Circuit

The considerations in this book have been concerned with establishing and employing the analogies between electrical, mechanical and acoustical systems. As an extension, analogies between electrical and mag-

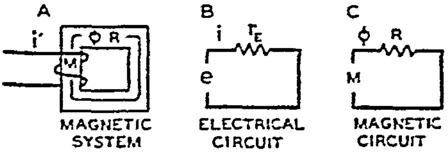


FIG. 15.1. The magnetic system, electrical circuit and magnetic circuit of a magnetic circuit of a single material. A. The magnetic system: i', the electrical current in the coil. M, the magnetomotive force developed because of the current in the coil. ϕ, the total magnetic lines in the magnetic circuit. R, reluctance of the magnetic circuit. B. The electrical circuit analogous to the magnetic circuit: e, the electromotive force analogous to magnetomotive force. i, the current analogous to total magnetic lines. r_E, the electrical resistance analogous to reluctance. C. The magnetic circuit of the magnetic system: M, the magnetomotive force. ϕ, the total magnetic lines. R, the reluctance.

netic quantities may be used to solve problems in magnetic systems by reducing the system to an analogous electrical network. The performance of the network may then be determined by electrical circuit theory.

A simple magnetic system consists of a closed loop of magnetic material encircled by a coil of wire carrying a current. The fundamental equation of the magnetic circuit of Fig. 15.1*A* is given by

$$\phi = \frac{M}{R} \qquad 15.1$$

where ϕ = total lines, in maxwells,
M = magnetomotive force generated by a current in the coil, in gilberts, and
R = reluctance, in gilberts per maxwell.

Equation 15.1 is analogous to Ohm's law in electrical circuits and may be expressed as follows:

$$i = \frac{e}{r_E} \qquad 15.2$$

where i = current, in abamperes,
e = electromotive force, in abvolts, and
r_E = electrical resistance, in abohms.

The electrical circuit of Fig. 15.1*B* is analogous to the magnetic system of Fig. 15.1*A*. The magnetic circuit of the magnetic system is shown in Fig. 15.1*C*.[1]

TABLE 15.1

Electrical			Magnetic		
Quantity	Unit	Symbol	Quantity	Unit	Symbol
Electromotive Force	Volts $\times 10^8$	e	Magnetomotive Force	Gilberts	M
Electrical Resistance	Ohms $\times 10^9$	r_E	Reluctance	Gilberts per Maxwell	R
Current	Amperes $\times 10^{-1}$	i	Flux	Maxwells	ϕ

[1] As in the case of establishing analogies between electrical, mechanical and acoustical systems, the elements of the electrical network analogous to the magnetic system have been labeled with electrical symbols. However, by following the procedure outlined in the footnotes on pages 131 and 188, the conventional procedure will be to label the analogous electrical network with the magnetic symbols as shown in Fig. 15.1*C*, Fig. 15.4*C* and so forth. It is literally accurate to label this network with the caption "analogous electrical network of the magnetic system." For the sake of brevity these networks will be labeled "magnetic network." Where there is only one path, "circuit" will be used instead of "network."

The analogous quantities and symbols in electrical and magnetic systems are shown in Table 15.1. The data of Table 15.1 summarize the discussion relating to equations 15.1 and 15.2.

The magnetomotive force, developed by a current in the coil in the electromagnetic system as shown in Fig. 15.1, is given by

$$M = 4\pi n i' \qquad 15.3$$

where M = magnetomotive force, in gilberts,
n = number of turns in the coil, and
i' = current in the coil, in abamperes.

The other quantity in equation 15.1 is the reluctance of the magnetic circuit. The magnetic circuit of Fig. 15.1 consists of a magnetic conductor of ferromagnetic material. With no leakage assumed, the reluctance of the magnetic conductor shown in Fig. 15.2 is given by

$$R = \frac{l}{\mu A} \qquad 15.4$$

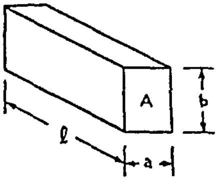


Fig. 15.2. Magnetic conductor. l, the length. a, the width. b, the height. $A = a \times b$. A, the cross-sectional area of the magnetic conductor.

where R = reluctance, in gilberts per maxwell,
l = length of the conductor, in centimeters,
A = cross-sectional area of the magnetic conductor, in square centimeters, and
μ = permeability of the magnetic material.

The electrical resistance of a conductor is given by

$$r_E = \frac{l}{\delta A} \qquad 15.5$$

where r_E = electrical resistance, in abohms,
l = length of the conductor, in centimeters,
A = cross-sectional area of the electrical conductor, in square centimeters, and
δ = conductivity of the electrical conductor, in abmhos.

By comparing equations 15.4 and 15.5 it will be seen that the formulas for magnetic reluctance and electrical resistance for magnetic and electrical conductors are analogous. Furthermore, magnetic permeability is analogous to electrical conductivity.

The permeability of a magnetic material can be obtained from the relation

$$\mu = \frac{B}{H} \qquad 15.6$$

where B = flux density, in gausses, and
H = magnetizing force, in oersteds.

The B-H versus B characteristic for various magnetic materials[2,3] is shown in Fig. 15.3. The permeability of the ferromagnetic material can be obtained from equation 15.6.

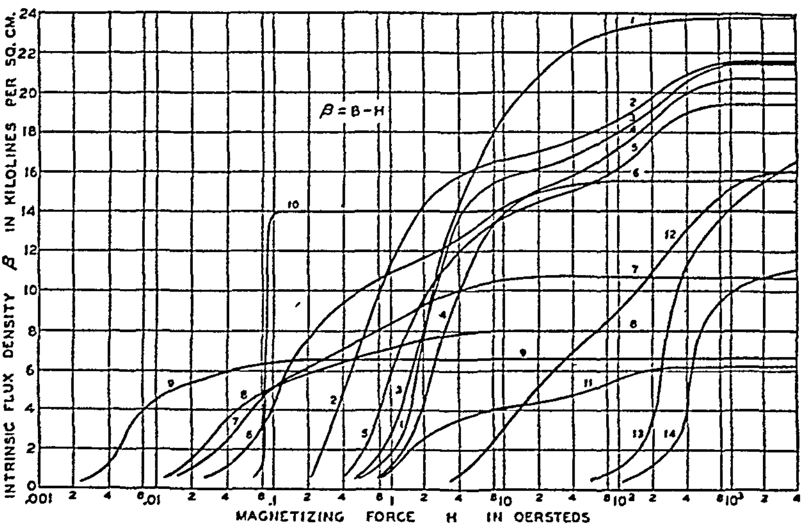


FIG. 15.3. D-C magnetization characteristics of various materials. 1. Permandur, 50 per cent cobalt and 50 per cent iron. 2. Electrolytic iron. 3. Armco iron. 4. Annealed cold drawn steel. 5. Medium hard silicon steel. 6. Nickel iron alloy, 47 per cent nickel. 7. Permalloy, 79 per cent nickel. 8. Allegheny, Mumetal. 9. Supermalloy. 10. Deltamax. 11. Pure nickel. 12. Cast iron. 13. Cobalt, permanent magnet steel. 14. Alnico, permanent magnet alloy. After Kentner, Bozorth.

The magnetic circuit of Fig. 15.1*C* can be solved by means of equations 15.1 to 15.4 inclusive and 15.6, and the data of Fig. 15.2.

[2] Kentner, A. E., *Gen. Elec. Rev.*, Vol. 45, No. 11, p. 633, 1942.
[3] Bozorth, "Ferromagnetism," D. Van Nostrand Co., Princeton, N. J., 1951.

15.4. Series Magnetic Circuit of Two Magnetic Materials

A magnetic system consisting of a circuit of two magnetic materials in series is shown in Fig. 15.4*A*. The analogous electrical circuit of the

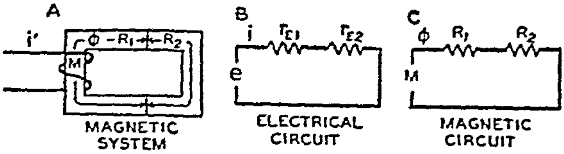


FIG. 15.4. The magnetic system, electrical circuit and magnetic circuit of a magnetic circuit of two different materials. A. *The magnetic system: i', the current in the coil.* M, the magnetomotive force developed because of the current in the coil. ϕ, the total magnetic lines in the magnetic circuit. R_1 and R_2, the reluctances of the two parts of the magnetic circuit. B. *The electrical circuit analogous to the magnetic circuit:* e, the electromotive force analogous to the magnetomotive force. i, the current analogous to the total magnetic lines. r_{E1} and r_{E2}, the electrical resistances analogous to the reluctances. C. *The magnetic circuit of the magnetic system:* M, the magnetomotive force. ϕ, the total lines. R_1 and R_2, the reluctances in the magnetic circuit.

magnetic system is shown in Fig. 15.4*B*; the magnetic circuit of this system is shown in Fig. 15.4*C*. The magnetic flux in the magnetic circuit is given by

$$\phi = \frac{M}{R_1 + R_2} \qquad 15.7$$

where ϕ = total lines, in maxwells,
M = magnetomotive force developed by the current in the coil, in gilberts,
R_1 = reluctance of the magnetic path 1, in maxwells per gilbert, and
R_2 = reluctance of the magnetic path 2, in maxwells per gilbert.

15.5. Magnetic Network

A magnetic system consisting of a magnetic network is shown in Fig. 15.5*A*. The analogous electrical network of the magnetic system is shown in Fig. 15.5*B*; the magnetic network of this system is shown in Fig. 15.5*C*. The magnetic flux in the magnetic path 1 is given by

$$\phi_1 = \frac{M(R_2 + R_3)}{R_1R_2 + R_1R_3 + R_2R_3} \qquad 15.8$$

where ϕ_1 = total lines, in maxwells, in magnetic path 1,
M = magnetomotive force developed by the current in the coil, in gilberts,
R_1 = reluctance of the magnetic path 1, in gilberts per maxwell,
R_2 = reluctance of the magnetic path 2, in gilberts per maxwell, and
R_3 = reluctance of the magnetic path 3, in gilberts per maxwell.

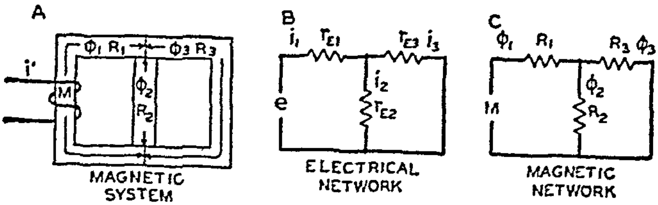


FIG. 15.5. A magnetic system, electrical network and magnetic network of a magnetic network consisting of three circuits. A. The magnetic system: i', the current in the coil. M, the magnetomotive force developed because of the current in the coil. ϕ_1, ϕ_2 and ϕ_3, the total lines in the three magnetic circuits. R_1, R_2 and R_3, the reluctances in the three magnetic circuits. B. The electrical network analogous to the magnetic circuit: e, the electromotive force analogous to the magnetomotive force. i, the current analogous to the total magnetic lines. r_{E1}, r_{E2} and r_{E3}, the electrical resistances analogous to the reluctances. C. The magnetic network: M, the magnetomotive force. ϕ_1, ϕ_2 and ϕ_3, the total lines in three magnetic circuits. R_1, R_2 and R_3, the reluctances in the three magnetic circuits.

The magnetic flux in magnetic path 2 is given by

$$\phi_2 = \frac{MR_3}{R_1R_2 + R_1R_3 + R_2R_3} \qquad 15.9$$

where ϕ_2 = total lines, in maxwells in magnetic path 2.

The magnetic flux in magnetic path 3 is given by

$$\phi_3 = \frac{MR_2}{R_1R_2 + R_1R_3 + R_2R_3} \qquad 15.10$$

where ϕ_3 = total lines, in maxwells, in path 3.

15.6. Magnetic Circuit with an Air Gap

A magnetic circuit with an air gap is shown in Fig. 15.6. The reluctance of an air gap is given by

$$R_1 = \frac{l}{A} \qquad 15.11$$

where R_1 = reluctance, in gilberts per maxwell,
l = length of the air gap in the direction of the flux, in centimeters, and
A = area of the air gap, in square centimeters.

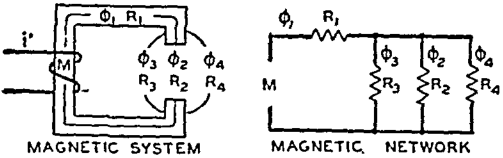


FIG. 15.6. The magnetic system and magnetic network of a magnetic circuit with an air gap. In the magnetic system and magnetic circuit: i', the current in the coil. M, the magnetomotive force due to the current in the coil. ϕ_1, the magnetic lines in the iron circuit. ϕ_2, the magnetic lines in the air gap. ϕ_3 and ϕ_4, the magnetic lines due to leakage. R_1, the reluctance of the iron circuit. R_2, the reluctance of the air gap. R_3 and R_4, the reluctances of the leakage paths.

The magnetic permeability of the air gap is unity.

In the system of Fig. 15.6 there are four paths, namely, the iron path 1, the air-gap path 2 and the magnetic-leakage paths 3 and 4. The air-gap flux is given by

$$\phi_2 = \frac{MR_3R_4}{R_1(R_2R_3 + R_2R_4 + R_3R_4) + R_2R_3R_4} \qquad 15.12$$

where ϕ_2 = total lines, in maxwells, in the magnetic path 2 and
R_1, R_2, R_3 and R_4 = reluctances of the magnetic paths 1, 2, 3 and 4, in gilberts per maxwell.

The leakage reluctance can be computed by approximations as depicted in Fig. 15.7. The reluctance of the incremental lengths and the incremental areas may be computed approximately as follows: The reluctance of the air path between the area A_2 and A_3 is given by

$$R_{23} = \frac{2l_{23}}{(A_2 + A_3)} \qquad 15.13$$

where R_{23} = reluctance between the areas A_2 and A_3, in gilberts per maxwell,
l_{23} = length of the magnetic path between the areas A_2 and A_3, in centimeters, and
A_2 and A_3 = cross-sectional areas at the ends of the path l_{23}, in square centimeters.

By following the computation of the magnetic path l_{23} the other paths l_{12} and l_{34} may be computed. The resultant reluctance is the sum of the reluctances of the paths in series as follows:

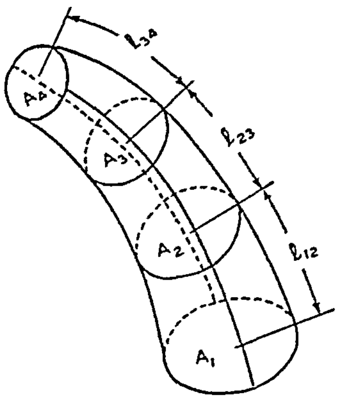


FIG. 15.7. Magnetic circuit with variable elements of cross-sectional area and variable elements of length.

$$R_{T1} = R_{12} + R_{23} + R_{34} \qquad 15.14$$

where R_{T1} = total reluctance of the paths l_{12}, l_{23} and l_{34}, and

R_{12}, R_{23} and R_{34} = reluctances of the paths l_{12}, l_{23} and l_{34}.

In the case of a number of paths of the type shown in Fig. 15.7 in parallel, the resultant reluctance of the parallel paths is given by

$$\frac{1}{R_T} = \frac{1}{R_{T1}} + \frac{1}{R_{T2}} + \frac{1}{R_{T3}} \cdots, \qquad 15.15$$

Where the reluctance R_{T1} is given by equation 15.14, and R_{T2}, R_{T3} and so forth are reluctances of other parallel magnetic flux paths determined in the same manner as that employed in equation 15.14.

15.7. Magnetic Circuit with a Permanent Magnet

The magnetic system of Fig. 15.8 is the same as that of Fig. 15.6 save

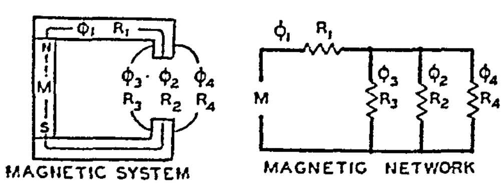


FIG. 15.8. The magnetic system and magnetic network of a magnetic circuit with an air gap and energized by a permanent magnet. In the magnetic system and magnetic circuit: M, the magnetomotive force of the permanent magnet. ϕ_1, the magnetic lines in the iron circuit. ϕ_2, the magnetic lines in the air gap. ϕ_3 and ϕ_4, the magnetic lines due to leakage. R_1, the reluctance of the iron circuit. R_2, the reluctance of the air gap. R_3 and R_4, the reluctances of the leakage paths.

that a permanent magnet is used instead of the electromagnet. The magnetomotive force developed by a permanent magnet is given by

$$M = Hl \qquad 15.16$$

where M = magnetomotive force, in gilberts,
H = demagnetizing force, in oersteds, and
l = length of the permanent magnet.

The demagnetizing force H can be obtained from the characteristics of Fig. 15.9. The induction in gausses represents the magnetic flux

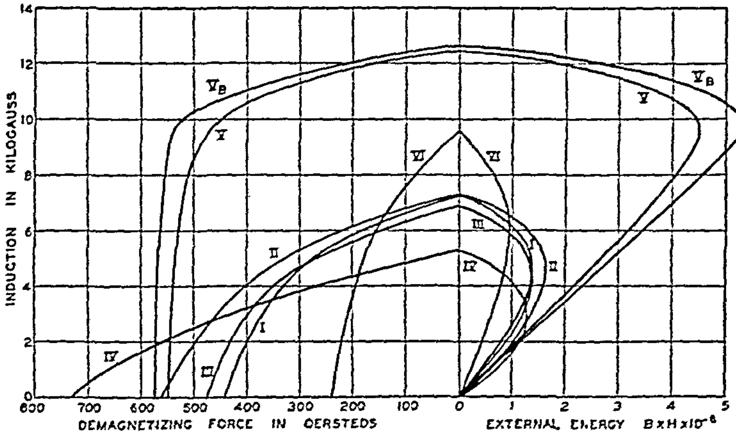


FIG. 15.9. Demagnetization and energy characteristics of permanent magnet materials. I. Alnico I. II. Alnico II. III. Alnico III. IV. Alnico IV. V. Alnico V. VB. Alnico VB. VI. 36 per cent cobalt.

density in the permanent magnet. The average flux density in gausses in the magnet is the total flux ϕ_1, in maxwells, divided by the cross-sectional area of the magnet in square centimeters. The demagnetizing force H, in oersteds, corresponding to this flux density can be obtained from Fig. 15.9. The magnetomotive force M, in gilberts, supplied by the permanent magnet is the product of the demagnetizing force H, in oersteds, and the length of the magnet in centimeters.

15.8. Balanced Armature Magnetic System

A balanced armature magnetic system of the type shown in Fig. 15.10 is used for the transducer in the magnetic microphone or the magnetic phonograph pickup. When the magnetic flux through the coil surrounding the armature varies with respect to time, a voltage is induced

in the coil. Therefore, the important consideration is the magnetic flux through the armature. The reluctance of the iron path is very small

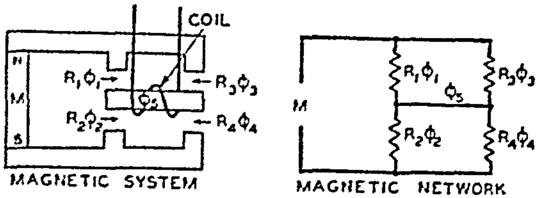


FIG. 15.10. The magnetic system, coil and magnetic network of a balanced armature transducer. In the magnetic system and magnetic network: M, the magnetomotive force of the permanent magnet. ϕ_1, ϕ_2, ϕ_3 and ϕ_4, the magnetic lines in the air gaps. R_1, R_2, R_3 and R_4, the reluctances of the air gaps. ϕ_5, the magnetic lines through the armature.

and may be neglected. Under these conditions the only reluctances under consideration are those representing the air gaps. The magnetic flux through the armature is given by

$$\phi_5 = \frac{M(R_2R_3 - R_1R_4)}{(R_1 + R_2)R_3R_4 + (R_3 + R_4)R_1R_2} \qquad 15.17$$

where ϕ_5 = total lines through the armature, in maxwells, and

R_1, R_2, R_3 and R_4 = reluctances of the air gaps in Fig. 15.10, in gilberts per maxwell.

Referring to equation 15.17, it will be seen that if $R_2R_3 = R_1R_4$, there will be no flux through the armature. This is the balanced condition. However, if the armature is rotated about the point 0, there will be a magnetic flux in the armature determined by the difference of R_2R_3 and R_1R_4.

15.9. Practical Considerations

In general, in the design of magnetic structures considerable experimental work is required because in most cases it is very difficult to predict the performance by theoretical analysis alone. For example, measurements of the air-gap flux density, the leakage flux and the flux density in various parts of the magnetic circuit can be made by means of a calibrated fluxmeter and a loop or coil. The air-gap flux density

can be obtained by means of a calibrated coil and fluxmeter. The flux in any part of the magnetic circuit can be obtained by placing a loop of one or more turns around the section to be tested, the loop being connected to the fluxmeter. This coil is then pulled out to a point where there is no flux. The flux can be obtained from the deflection and calibration of the fluxmeter and the number of turns in the coil. From these measurements, data can be obtained which will give the total flux, the leakage flux, the air-gap flux and the flux density in the iron and permanent magnet structure. These data together with the equations and data in the preceding considerations will indicate the direction of improvement from the standpoint of air-gap flux density, leakage flux and optimum cross section of the iron and permanent magnet.

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